

HEAT KERNELS ON EUCLIDEAN COMPLEXES

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by

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In this thesis we describe a type of metric space called an Euclidean polyhedral complex. We define a Dirichlet form on it; this is used to give a corresponding heat kernel. We provide a uniform small time Poincaré inequality for complexes with bounded geometry and use this to determine uniform small time heat kernel bounds via a theorem of Sturm. We then consider such complexes with an underlying finitely generated group structure. We use techniques of Saloff-Coste and Pittet to show a large time asymptotic equivalence for the heat kernel on the complex and the heat kernel on the group.

BIOGRAPHICAL SKETCH

Melanie Pivarski was born on August 13, 1977 somewhere in the outskirts of Pittsburgh to Lynn and Thomas Pivarski. She grew up with her parents and two sisters, Kara and Janelle; her grandmother, Caroline Matovick, lived a few blocks away. They can all attest to the fact that yes, Melanie has always talked with her hands.

She attended Colfax Elementary School where she greatly enjoyed Ms. Kengor's math classes; partly, this was because she could spend time doing logic puzzles. She was also involved in Girl Scouts with Chrissy. She then attended Springdale High School where she participated in many different activities, most notably art classes, Drama Club with Ms. Frauenholz, and Academic Games.

From 1995 through 1999, she attended Carnegie Mellon University, where she majored in mathematics and minored in computer science. While there, she took Prof. Mizel's freshman analysis course out of Apostol. This class convinced her that she needed to study mathematics. In her analysis class she met Helena and Ruth who became her good friends and study partners. Computers are a part of the culture at CMU, and so she found herself in a number of fun computer science classes. Though she spent much of her time on math and computers, she found time to take some ballet classes with her friend Robert and some introductory Polish classes at the University of Pittsburgh. Quite significantly, she met Jim McCann (now Jim McCann Pivarski) during freshman orientation. They began dating that fall and were married in June of 1999.

Melanie and Jim moved to Ithaca in the summer of 1999, where they became graduate students in math and physics respectively. Melanie has greatly enjoyed her time in the math department; while there she was involved in the teaching seminars, the outreach program Expanding Your Horizons, the women in math

potlucks, the 120A Cafe, and the Gingerbread House contests, specializing in tower constructions. She also studied some math.

Outside the math department, Melanie found many things to do. She's taken ballet classes and sang in the choir at St. Catherine's for much of her time here. She's also been involved in the St Catherine's young adult group, the PreCana team, and spent a few years helping out at Loaves and Fishes. Through all of this, she's met a number of interesting people and had a variety of experiences. She considers herself to be more mature than she's ever been before, and she hopes to continue growing and learning throughout her life.

In the fall of 2006, Melanie and Jim will move to College Station where they will be employed as postdocs at Texas A & M University.

To Friendship! And most especially to Jim!

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LIST OF SYMBOLS

- X the complex
- $X^{(k)}$ the k -skeleton of X
- γ a path in X
- $L(\gamma)$ length of γ
- $d_X(\cdot, \cdot)$ distance in X induced by the Euclidean metric
- $d_{X^{(i)}}(\cdot, \cdot)$, $d_i(\cdot, \cdot)$ distance in $X^{(i)}$ induced by the Euclidean metric
- μ measure on X
- $E(\cdot, \cdot)$ energy form constructed via Γ -limit
- $\mathcal{E}(\cdot, \cdot)$ energy form constructed via gradients
- Δ , Δ^k Laplacian for X , $X^{(k)}$
- Δ^Ω Laplacian for $\Omega \subset X$ with Dirichlet boundary condition
- ∇f , F gradient of f
- $d\mu(x)$, dx equivalent ways of writing the differential form
- $\text{Lip}(X)$ the set of Lipschitz functions on X
- $W^{1,p}(X)$ the Sobolev space on X of functions in $L^p(X)$ with first derivatives in L^p
- $C_0^{\text{Lip}}(X)$ the set of compactly supported Lipschitz functions on X
- f_B average integral over the set B
- I_Ω indicator function on Ω (1 if in Ω , 0 if not)
- f_E average value of f on the set E
- ∂E the boundary of the set E
- $S^{(k)}$ the k -sphere
- W_k , $W_{j,k}$ wedges of a ball in X
- $N(j)$ the list of indices of faces adjacent to W_j including j
- M degree bound on X
- ℓ lower bound on edge lengths of X

- α smallest interior angle in X
- κ constant related to multiples of radii
- R_0 bound on radius defined in terms of κ
- C_{Weak} constant in weak Poincaré inequality
- C_P constant for the Poincaré inequality
- \mathcal{F} collection of balls in the Whitney cover
- r_B radius of the ball B
- B_z the central ball in the Whitney cover
- $\mathcal{F}(B)$ a string of balls that takes B_z to B
- $h_t(x, y)$, $h_t^k(x, y)$ heat kernel on X , $X^{(k)}$
- H_t heat semigroup on X
- τ exit time for Brownian motion on a set
- X_t random variable for location of a Brownian motion in a subset of X
- $h_t^\Omega(x, y)$ heat kernel on $\Omega \subset X$
- H_t^Ω heat semigroup on $\Omega \subset X$
- $p_t(x, y)$, $p_t^A(x, y)$ heat kernel on G , $A \subset G$
- K^n , K_A^n n step transition operator on G , $A \subset G$
- $\lambda_\Omega(i)$ i th eigenvalue for H_1^Ω
- $\beta_A(i)$ i th eigenvalue for K_A^1
- $\text{Tr}(K^n)$ trace of K^n
- G group
- S generating set for G
- Y compact subset of X with the property $X/Y = G$
- $|g|$ word length of $g \in G$
- $d_G(\cdot, \cdot)$ distance in G with respect to word length
- $d_Y(\cdot, \cdot)$ distance in Y based on Euclidean paths in Y .

- $\text{diam}(Y)$ diameter of Y with respect to d_Y
- $Y^{(i)}$ the i skeleton of Y ; $X^{(i)} \cap Y$
- $|A|$, $\#A$ number of elements in A
- Vol , Vol_X , Vol_G volume (with respect to X , G)
- B_r , $B_X(r)$, $B_G(r)$ ball of radius r (in X , G)
- $B_X(x, r)$, $B_G(g, r)$ ball in X (G) with radius r centered at x (g)
- $E(f, f)$ energy form; for G this is $\frac{1}{|S|} \sum_{g \in G} \sum_{s \in S} |f(g) - f(gs)|^2$, for X this is $-\langle \nabla f, \nabla f \rangle$
- $|\nabla f(x)|$ length of gradient; on G this is $\sqrt{\frac{1}{|S|} \sum_{s \in S} |f(x) - f(xs)|^2}$.
- $\|f\|_{p,A}$ the L^p norm restricted to a subset A ; for G this is $(\sum_{x \in A} |f(x)|^p)^{1/p}$, for X this is $(\int_A |f(x)|^p dx)^{1/p}$
- $\{\gamma_i\}_{i=1}^N$ centers of the balls of radius δ covering Y
- $\{g\gamma_i\}_{i=1..N; g \in G}$ centers of the balls of radius δ covering X
- $C_{N,S}$ maximum number of balls overlapping a point in X
- $\text{group } f$ a new function from $(G, N) \rightarrow R$ defined to be $\int_{B_X(g\gamma_i, \delta)} f(x) dx$
- C_{XG} , C_0 constants used to compare metrics in G and X
- C_H constant for the Harnack inequality
- χ , χ_g a smooth function on X , χ translated by g
- C_{sup} constant used to bound the support of χ
- C_{Over} constant bound on the maximum number of χ_g supported at any point in X
- C_g bound on the magnitude of the gradient of χ
- $\text{comp } f(x)$ a new function from $X \rightarrow R$ defined by $\sum_{g \in G} f(g) \chi_g(x)$
- C_{grad} constant bound comparing norms of gradients
- $\lfloor x \rfloor$ the largest integer less than or equal to x
- $\lceil x \rceil$ the smallest integer greater than or equal to x
- $C_0(\Omega)$ continuous functions which are compactly supported in Ω
- $U(A)$ subset of X depending on A

Chapter 1

Setup for the Complexes

1.1 Introduction

We will study how local and global geometries affect heat kernels on a set of metric spaces called Euclidean polyhedral complexes.

Euclidean complexes are formed by taking a collection of n dimensional convex polytopes and joining them along $n - 1$ dimensional faces. Within each polytope, we will have the same metric structure as R^n . When we join them, we will glue the faces of two polytopes together so that points on one face are identified with points on the other face, and the metrics on those faces are preserved. We will require that these structures have a countable number of polytopes, are locally finite, and have lower bound on the interior angles and edge lengths. The complex formed by looking at k dimensional faces is called the k -skeleton. For instance, the 0-skeleton is set of vertices. A 1-skeleton is a graph where the space includes both vertices and points on the edges; sometimes this is called a metric graph [24]. Note that we can triangulate any convex polytope to obtain a collection of simplices, and so this structure is essentially equivalent to looking at a simplicial complex.

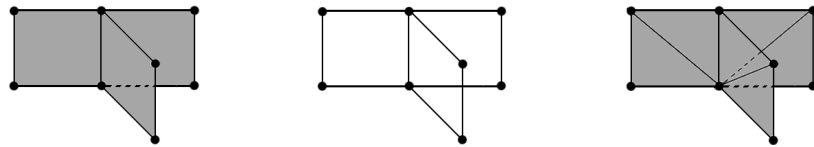


Figure 1.1: Example of a 2 dimensional Euclidean Complex (left), its 1-skeleton (center), one possible triangulation (right).

Let $h_t^k(x, y)$ be the heat kernel on the k -skeleton. This is the fundamental

solution to the heat equation $\partial_t u - \Delta u = 0$ on the k -skeleton. It can be used to describe the probability that we travel from x to y in time t when our movement is restricted to the k -skeleton.

Theorem 1.1.1. *For X a uniformly locally finite Euclidean complex of dimension n whose interior angles and edge lengths are bounded below. Fix $T \in (0, \infty)$. There exist $c, C \in (0, \infty)$ such that for any $x \in X$ and $t < T$ we have:*

$$\frac{c}{t^{k/2}} \leq h_t^k(x, x) \leq \frac{C}{t^{k/2}}.$$

Note that this claims that the heat kernel on the k -skeleton behaves, up to a constant that is independent of where in X we are, like the heat kernel on R^k asymptotically when $t \rightarrow 0$. The local behavior reflects the local geometry and structure of our space.

Theorems in Sturm [30] can be applied to Euclidean complexes to show that on any compact subset of $X^{(k)}$, the heat kernel is locally like the one on R^k , with constants that depend on the choice of compact subset. The essential difference in our theorem is that the constants are uniform throughout the entire complex.

An interesting example of these complexes comes from biology. In a paper by Billera, Holmes, and Vogtmann [3] they describe a way of classifying distances between phylogenetic trees, which are trees that describe evolution of species. One can form an Euclidean complex, where each of the faces corresponds to a different tree, and one moves through the points in the face by changing the edge lengths in the tree. One can then consider probability distributions on this space to determine likely genetic ancestry.

Euclidean complexes are also examples of fractal “blow-ups”, which are infinite fractals that are locally nice but globally have a structure with repetition. See

Kigami [22] for a description of these fractals. In this setting, our small time asymptotic estimates apply. Note that these examples need not be compact. In [1], Barlow and Kumagai studied the small time asymptotic of heat kernels for compact self-similar sets.

Another collection of examples can be found by considering metric spaces, X , which are acted upon by a finitely generated group, G of isometries. When we take the space and mod out by that group, we obtain a compact set $Y = X/G$. When Y can be expressed as a finite Euclidean complex, then X is an Euclidean complex as well. Note that the k -skeleton of Y will be the $(k$ -skeleton of $X)/G$. A simple example of this is $X = \mathbf{R}^2$, Y = the unit square, and $G = \mathbf{Z}^2$. A more interesting example occurs when G is the free group; there the space is globally hyperbolic, but locally Euclidean. With this added group structure, we can describe the large time behavior of the heat kernel. We write the heat kernel on a group as $p_t(\cdot, \cdot)$.

Proposition 1.1.2. *Let X be a locally finite countable Euclidean complex of dimension n and let G be finitely generated group G . If X/G is a complex comprised of a finite number of polytopes with Euclidean metric, we have:*

$$p_t(x, x) \simeq h_t(x, x) \text{ as } t \rightarrow \infty.$$

Our main result says that, up to a constant, the heat kernel will behave the same asymptotically as $t \rightarrow \infty$ on both the group and the complex. By transitivity, it will behave the same asymptotically regardless of which k -skeleton we consider.

This theorem relates to a paper of Pittet and Saloff-Coste [26]. They show that a manifold M which has a finitely generated group of isometries G satisfies $\sup_x h_t^M(x, x) \simeq h_t^G(e, e)$ for large t .

In chapter one, we describe our set-up. We provide definitions for the complex and skeletons and then define an energy form and a Laplacian on them. In chapter

two, we will prove the initial theorem by first showing that a series of Poincaré inequalities hold, starting with one for balls where the radius of the ball depends on the center and generalizing until the result is uniform in space. In chapter three, we apply these inequalities to a result of Sturm [29] to yield a small time on diagonal heat kernel asymptotic with a uniform constant. We also provide off diagonal estimates with constants that depend on $d(x, y)$, but not on where x and y are located. We give several examples of heat kernels. In chapter four, we consider complexes with underlying group structure. We describe how to compare metrics on the complex and those on the underlying group, as well as how to switch from a function on a group to one on a complex and vice versa. We then use the metric comparison as well as our small time Poincaré inequality to compare norms of functions on complexes and their group counterparts. In chapter five, we consider heat kernels on the group and the complex. We split into two cases; nonamenable groups, which have exponentially fast heat kernel decay, and amenable groups. For the amenable groups, we look at heat kernels restricted to subsets of our space, and then take a Følner sequence to limit to a bound on the heat kernels themselves. In this way, we prove the second theorem.

1.2 Geometry of the Complexes

We will take our definitions of polytopes and polyhedral sets from Grünbaum's Convex Polytopes [16].

Definition 1.2.1. A polyhedron K is a subset of R^n formed by intersecting a finite family of closed half spaces of R^n . Note that this can be an unbounded set, but it will be convex.

Definition 1.2.2. A set F is a face of K if $F = \emptyset$, $F = K$, or if $F = H \cap K$ where H is a supporting hyperplane of K . H is a supporting hyperplane of K if $H \cap K \neq \emptyset$ and H does not cut K into two pieces.

Definition 1.2.3. A point $x \in K$ is an extreme point of a set K if the only $y, z \in K$ which are solutions to $x = \lambda y + (1 - \lambda)z$ for some $\lambda \in (0, 1)$ are $x = y = z$. That is, x cannot be expressed as a convex combination of points in $K - \{x\}$. Note that the extreme points of K are faces for K .

Definition 1.2.4. A polytope is a compact convex subset of R^n which has a finite set of extreme points. This is equivalent to saying it is a bounded polyhedron.

Definition 1.2.5. A polyhedral complex X is the union of a collection, \mathcal{X} , of convex polyhedra which are joined along lower dimensional faces. By this we mean that for any two distinct polyhedra $P_1, P_2 \in \mathcal{X}$,

- $P_1 \cap P_2$ is a polyhedron whose dimension satisfies $\dim(P_1 \cap P_2) < \max(\dim(P_1), \dim(P_2))$ and
- $P_1 \cap P_2$ is a face of both P_1 and P_2 . We allow this face to be the empty set.

We do not have a specific embedding for the complex, X ; however, we require each polyhedra to have a metric which is consistent with that of its faces.

Note that this definition implied $P_1 \cap P_2$ is a connected set. This rules out expressing a circle as two edges whose ends are joined, but it allows us to write it as a triangle of three edges. This is not very restrictive, as we can triangulate the polyhedra in order to form a complex which avoids the overlap.

Simplicial complexes are an example of a polyhedral complex; the difference here is that we allow greater numbers of sides. Note that we allow infinite polyhedra, not just finite polytopes.

Definition 1.2.6. Define a p -skeleton, $X^{(p)}$, for $0 \leq p \leq \dim X$ to be the union of all faces of dimension p or smaller. Note that this is also a polyhedral complex.

Definition 1.2.7. A maximal polyhedron is a polyhedron that is not a proper face of any other polyhedron. The set of maximal polyhedra of X is denoted \mathcal{X}_{MAX} . We say X is dimensionally homogeneous if all of its maximal polyhedra have dimension n . Note that in combinatorics literature this is called pure.

Definition 1.2.8. X is locally $(n-1)$ -chainable if for every connected open set $U \subset X$, $U - X^{(n-2)}$ is also connected. For a dimensionally homogeneous complex X this is equivalent to the property that any two n dimensional polyhedra that share a lower dimensional face can be joined by a chain of contiguous $(n-1)$ or n dimensional polyhedra containing that face.

Definition 1.2.9. We call X admissible if it is both dimensionally homogeneous and in some triangulation X is locally $(n-1)$ -chainable.

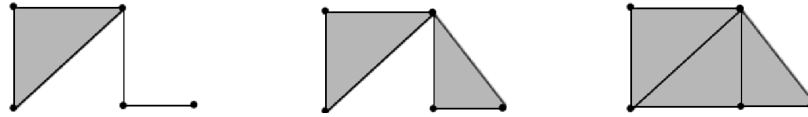


Figure 1.2: Examples of a complex which is not dimensionally homogeneous (left), one which is not 1-chainable (center), and one which is admissible (right).

We will be working with connected admissible complexes, and for our purposes, we'd like to consider polyhedra that have an Euclidean metric. Let X be an n -dimensional complex. When two polyhedra share a face, we require these metrics to coincide. For points x and y in different polyhedra, we define the distance as follows.

Definition 1.2.10. Consider the set of paths connecting x to y which consist of a finite number of line segments. We can label each of these by the points it crosses in the $(n - 1)$ skeleton. Set $\gamma = \{x = x_0, x_1, x_2, \dots, x_k = y\}$ where $x_i \in (n - 1)$ -skeleton for $i = 1..k - 1$, and x_i, x_{i+1} are both in the closure of the same maximal polyhedron. Then set $L(\gamma) = \sum_{i=1}^k d(x_{i-1}, x_i)$. We define the distance between points in different polyhedra to be $d(x, y) = \inf_{\gamma} L(\gamma)$.

Essentially, we are splitting the path into pieces, and letting the lengths of those pieces inside of the simplices be the standard lengths in R^n . Since our complex is created using closed polyhedra, if the geometry of the polyhedra is bounded, the inf will be realized. This will give us a length space; ie, one in which distances are realized by geodesics in the space. Discussions of length spaces and other metric measure spaces can be found in Heinonen [18] and Burago, Burago, and Ivanov [6].

Definition 1.2.11. Let $X = \cup_i P_i$, where the P_i are the maximal polyhedra. We will set the measure of A , a Borel subset of X , to be $\mu(A) = \sum_i \mu_i(A \cap P_i)$ where μ_i is the Lebesgue measure on P_i .

Notice that the measure within the interior of maximal polyhedra is the same as Lebesgue measure on R^n . This means that locally we will have all of the structure of R^n ; in particular, we will have volume doubling for balls contained in the interior of the maximal polyhedra. Since our complex is locally finite, volume doubling will hold locally for all points in the complex.

Definition 1.2.12. An admissible polyhedral complex, X , equipped with distance, $d(\cdot, \cdot)$ and measure μ is called an Euclidean polyhedral complex.

For brevity, we will often call this an Euclidean complex. A book which describes these complexes is Harmonic Maps Between Riemannian Polyhedra [11]. In it, the authors define these structures with a Riemannian metric and provide analytic results on both the complexes and functions whose domain and range are both complexes.

1.3 Analysis on the Complexes

1.3.1 The Dirichlet Form

Now that we've defined the space geometrically, we will define a Dirichlet form whose core consists of compactly supported Lipschitz functions.

Definition 1.3.1. A function f on a metric space X is called L -Lipschitz (alternately, Lipschitz) if there exists a constant $L \geq 0$ so that $d(f(x), f(y)) \leq Ld_X(x, y)$ for all x and y in X . The space of Lipschitz functions is denoted $\text{Lip}(X)$. The space of compactly supported Lipschitz functions is denoted $C_0^{\text{Lip}}(X)$.

Note that Lipschitz functions are continuous. By theorem 4 in section 5.8 of [12], for each $B_\epsilon(x) \subset X - X^{(n-1)}$ and $f \in C_0^{\text{Lip}}(X)$, f restricted to $B_\epsilon(x)$ is in the Sobolev space $W^{1,\infty}(B_\epsilon(x))$. This tells us that f has a gradient almost everywhere in $X - X^{(n-1)}$. Since $\mu(X^{(n-1)}) = 0$, f has a gradient for almost every x in X .

We would like an energy form that acts like $E(u, v) = \int_X \langle \nabla u, \nabla v \rangle d\mu$ with domain F to define our operator Δ with domain $\text{Dom}(\Delta)$. We can define this in a very general manner which does not depend on the local Euclidean structure by following a paper of Sturm [31]. We can also define it in a more straightforward manner which uses the geometry of X . We do both, and then show that they coincide.

Sturm assumes that the space (X, d) is a locally compact separable metric space, μ is a Radon measure on X , and that $\mu(U) > 0$ for every nonempty open set $U \subset X$. These assumptions hold both in our space, X , and on the skeletons, $X^{(k)}$. We begin by approximating E with a form E^r defined to be:

$$E^r(u, v) := \int_X \int_{B(x, r) - \{x\}} \frac{(u(x) - u(y))(v(x) - v(y))}{d^2(x, y)} \frac{2Nd\mu(y)d\mu(x)}{\mu(B_r(x)) + \mu(B_r(y))}$$

for $u, v \in \text{Lip}(X)$ where N is the local dimension. Note that whenever x is in a region locally like R^n , we have

$$\lim_{r \rightarrow 0} \frac{N}{\mu(B_r(x))} \int_{B(x, r) - \{x\}} \frac{(u(x) - u(y))^2}{d^2(x, y)} d\mu(y) = |\nabla u(x)|^2,$$

and so this form looks very similar to $E(u, u) = \int_X |\nabla u|^2 dx$.

This form with domain $C_0^{\text{Lip}}(X)$ is closable and symmetric on $L^2(X)$, and its closure has core $C_0^{\text{Lip}}(X)$. See Lemma 3.1 in [31]. One can take limits of these operators in the following way. The Γ -limit of the E^{r_n} is defined to be the limit that occurs when the following lim sup and lim inf are equal for all $u \in L^2(X, m)$. See Dal Maso[9] for a thorough introduction.

$$\Gamma - \limsup_{n \rightarrow \infty} E^{r_n}(u, u) := \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \inf_{\substack{v \in L^2(X) \\ \|u - v\| \leq \alpha}} E^{r_n}(v, v)$$

$$\Gamma - \liminf_{n \rightarrow \infty} E^{r_n}(u, u) := \lim_{\alpha \rightarrow 0} \liminf_{n \rightarrow \infty} \inf_{\substack{v \in L^2(X) \\ \|u - v\| \leq \alpha}} E^{r_n}(v, v).$$

For any sequence $\{E^{r_n}\}$ of these operators with $r_n \rightarrow 0$, there is a subsequence $\{r_{n'}\}$ so that the Γ -limit of $E^{r_{n'}}$ exists by Lemma 4.4 in [31]. These lemmas are put together into a theorem (5.5 in [31]) that tells us that this limit, E^0 , with domain $C_0^{\text{Lip}}(X)$ is a closable and symmetric form, and its closure, (E, F) , is a strongly local regular Dirichlet form on $L^2(X, m)$ with core $C_0^{\text{Lip}}(X)$.

Alternately, we can define the energy form using the structure of the space. We set $\mathcal{E}(\cdot, \cdot)$ to the following for $f \in C_0^{\text{Lip}}(X)$:

$$\mathcal{E}(f, f) = \sum_{X_M \in \mathcal{M}} \int_{X_M} |\nabla f|^2 d\mu(x).$$

Lemma 1.3.2. $\mathcal{E}(\cdot, \cdot)$ is a closable form. That is, for any sequence

$\{f_n\}_{n=1}^\infty \subset C_0^{\text{Lip}}(X)$ that converges to 0 in $L^2(X)$ and is Cauchy in $\|\cdot\|_2 + \mathcal{E}(\cdot, \cdot)$ we have $\lim_{n \rightarrow \infty} \mathcal{E}(f_n, f_n) = 0$.

Proof. To show this, we will first look at what happens on one fixed polyhedron, and then look at what happens on a complex. Let X_M be a maximal polyhedron. Since $\{f_n\}_{n=1}^\infty$ is Cauchy in the norm, we have

$$\lim_{m, n \rightarrow \infty} \left(\int_{X_M} (f_n - f_m)^2 d\mu \right)^{\frac{1}{2}} + \left(\int_{X_M} (\nabla f_n - \nabla f_m)^2 d\mu \right)^{\frac{1}{2}} = 0.$$

This gives us two functions, f and F which are the limits of f_n and ∇f_n respectively. We have $f = 0$ by assumption. We need to show that $F = 0$. For almost every $x, y \in X_M$ and line $\gamma_{x \sim y}$ in X_M we have

$$\int_{\gamma_{x \sim y}} \nabla f_n d\mu = f_n(y) - f_n(x).$$

Then we can take the limit as n goes to infinity to get

$$\lim_{n \rightarrow \infty} \int_{\gamma_{x \sim y}} \nabla f_n d\mu = \lim_{n \rightarrow \infty} f_n(y) - f_n(x) = 0.$$

This gives us $\lim_{n \rightarrow \infty} \nabla f_n(x) = 0$ for almost every $x \in X_M$.

Since the choice of X_M was arbitrary, this shows $\lim_{n \rightarrow \infty} \nabla f_n(x) = 0$ for almost every $x \in X$.

Showing L^2 convergence is a bit trickier, as we need to show that we can interchange the limit with the sum over the maximal polyhedra. We can do this

for $|\nabla f_n - \nabla f_m|$ by Fatou's Lemma.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sum_{X_M \in \mathcal{M}} \int_{X_M} |\nabla f_n|^2 d\mu &= \lim_{n \rightarrow \infty} \sum_{X_M \in \mathcal{M}} \int_{X_M} |\nabla f_n - \lim_{m \rightarrow \infty} \nabla f_m|^2 d\mu \\
&= \lim_{n \rightarrow \infty} \sum_{X_M \in \mathcal{M}} \int_{X_M} \lim_{m \rightarrow \infty} |\nabla f_n - \nabla f_m|^2 d\mu \\
&\leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{X_M \in \mathcal{M}} \int_{X_M} |\nabla f_n - \nabla f_m|^2 d\mu \\
&= 0.
\end{aligned}$$

This tells us that the form is closable. \square

We will show that the two energy forms are the same. To do this, we show that they are the same on the core $C_0^{\text{Lip}}(X)$; this gives equality on the domain.

Lemma 1.3.3. *Each function $f \in C_0^{\text{Lip}}(X)$ satisfies $E(f, f) = \mathcal{E}(f, f)$.*

Proof. We can write X as $(X - X^{(n-1)}) \cup X^{(n-1)}$; this is a collection of maximal polyhedra and a set of measure 0. The interior of the maximal polyhedra is a Riemannian manifold without boundary. X is also a locally compact length space, and so it satisfies the conditions of example 4G in [30]. This implies it has the strong measure contraction property with an exceptional set. Corollary 5.7 in [30] says that this then has $E(f, f) = \mathcal{E}(f, f)$ for each $f \in C_0^{\text{Lip}}(X)$. The equality is shown by approximating the forms using an increasing sequence of open subsets which limit to $X - X^{(n-1)}$. As $C_0^{\text{Lip}}(X)$ is a core for both E and \mathcal{E} , the Dirichlet forms are the same. \square

We will explain more clearly where the domain of this operator lies. The domain is the closure of $C_0^{\text{Lip}}(X)$ in the $W^{1,2}(X)$ norm. This domain is a subset of the set of functions which are in $W^{1,2}$ of the interiors of the maximal polyhedra.

Lemma 1.3.4. *For \mathcal{E} ,*

$$\overline{C_0^{\text{Lip}}(X)} \subset \overline{C(X) \cap \left(\bigcup_{X_M \in \mathcal{X}_{MAX}} \bigoplus W^{1,2}(X_M^o) \right)}$$

where the closure is taken with respect to the $W^{1,2}$ norm, $\|\cdot\|_2 + \mathcal{E}(\cdot, \cdot)$. X_M^o denotes the interior of X_M .

Proof. First note that $C_0^{\text{Lip}}(X) \subset C(X)$. For any $f \in C_0^{\text{Lip}}(X)$, we have the compact subset $Y = \text{supp}(f)$. Then f restricted to X_M^o will be in $W^{1,2}(X_M^o)$, since $\|\nabla f\|_{2, X_M^o} \leq \|\nabla f\|_{\infty, X_M} \mu(X_M^o \cap \text{supp}(f))$. This tells us

$$f \in \bigcup_{X_M \in \mathcal{X}_{MAX}} \bigoplus W^{1,2}(X_M^o).$$

We now have a containment without the closures:

$$C_0^{\text{Lip}}(X) \subset C(X) \cap \left(\bigcup_{X_M \in \mathcal{X}_{MAX}} \bigoplus W^{1,2}(X_M^o) \right).$$

As we then close both sides with respect to the same norm, we have:

$$\overline{C_0^{\text{Lip}}(X)} \subset \overline{C(X) \cap \left(\bigcup_{X_M \in \mathcal{X}_{MAX}} \bigoplus W^{1,2}(X_M^o) \right)}.$$

□

1.3.2 The Laplacian

The Dirichlet form uniquely determines a positive self-adjoint operator

$\{\Delta, \text{Dom}(\Delta)\}$ on L^2 where $F = \text{Dom}(\Delta^{\frac{1}{2}})$ and $E(u, v) = (u, \Delta v)$ for all $u \in F$ and $v \in \text{Dom}(\Delta)$. This is done by defining a collection of quadratic forms,

$E_\alpha(u, v) := E(u, v) + \alpha(u, v)$ for $\alpha > 0$. Then, by the Reisz representation theorem, there will be an operator G_α so that $E_\alpha(G_\alpha u, v) = (u, v)$ for any u, v in $\text{Dom}(E)$.

The set of these operators forms a C_0 resolvent. One can look at inverses, G_α^{-1} on

the image of G_α . We can then consider $\Delta = G_\alpha^{-1} - \alpha$ on the space $G_\alpha(\text{Dom}(E))$. One can show that this definition is independent of α . The domain of the operator is $\text{Dom}(\Delta) = G_\alpha(\text{Dom}(E))$. It's difficult to explicitly state exactly which functions are in $\text{Dom}(\Delta)$, but the domain is dense in $L^2(X)$. See Fukushima et al [14] for the full argument; a fine summary of this is done in Todd Kemp's lecture notes [20].

Note that this set-up will work on each of the skeletons, and so we can use it to define a different Laplacian on each of them. When we define the E^r on a k -skeleton, $X^{(k)}$, we'll set $N = k$, integrate over $X^{(k)}$, and let m be a k -dimensional measure. This technique will define Δ_k on a dense subset of $L^2(X^{(k)})$.

In the one dimensional case, this Laplacian gives us a structure called a quantum graph. Here, the functions in the domain of the Laplacian should be continuous and the inward pointing derivatives should sum to zero at each vertex. This is known as a Kirchoff condition. A nice introduction to these graphs and their spectra as well as a wide variety of references to the literature on them can be found in Kuchment [24].

In the two dimensional case, this operator is related to results in a paper of Brin and Kifer [5] which constructs Brownian motion on two dimensional Euclidean complexes. Bouziane [4] constructs and proves the existence of Brownian motion on admissible Riemannian complexes of any dimension. It would be interesting to determine whether these constructions define the same operator; however, that is not our focus.

Chapter 2

Local Poincaré Inequalities on X

In this chapter we will show that a uniform local Poincaré inequality holds for a certain class of admissible complexes. Local Poincaré inequalities have appeared in [33] and [11] for finite complexes or for compact subsets of complexes. In White's article [33], a global Poincaré inequality was shown for Lipschitz functions on an admissible complex made up of a finite number of polyhedra. The constant in this proof was linear in the number of polyhedra involved, and so it does not extend to an infinite complex. A uniform weak local inequality for Lipschitz functions was also shown on this finite complex. This too differs from our inequality in its dependence on a finite complex.

In Eells and Fuglede's book [11], they show that for any relatively compact subset of an admissible complex, a local Poincaré inequality will hold for locally Lipschitz functions with a constant that depends on the particular choice of compact subset. The larger complex itself can be infinite, but the constant in the inequality depends on our particular choice of compact subset.

We will show the following for $f \in \text{Lip}(X)$, under some assumptions on the geometry of X :

$$\|f - f_B\|_{p,B} \leq pP_0r\|\nabla f\|_{p,B}$$

where f_B is the average of f over B , $B = B(z, r)$, $r < R_0$. R_0 and P_0 are constants depending on the space, X .

Our result shows that a uniform local Poincaré inequality will hold for Lipschitz continuous functions on any ball of radius less than R_0 , where R_0 is fixed and depends only on the complex itself, not on the specific choice of ball. We require

our complex to be admissible. Our complex can be infinite, but we bound below the angles of the polyhedra and the distance between two vertices. We also bound above the number of polyhedra that join at a vertex. In both White and Eells and Fuglede these assumptions hold because their sets are either finite or relatively compact.

Connections between Poincaré inequalities and other analytic inequalities can be found in Sobolev met Poincaré by Hajlasz and Koskela [17].

2.1 Weak Poincaré Inequalities

We would like to prove a local Poincaré inequality for an admissible Euclidean polytopal complex. If we look at a convex subset of Euclidean space, this is a well known statement. We will show it first in a convex space, and then we will generalize it to our locally nonconvex space.

A note on our notation: often we will abbreviate $d\mu(x)$ by dx . Similarly, we will write the average integral of f over a set A by $\int_A f dx$.

Lemma 2.1.1. *Let Ω be a connected convex set with Euclidean distance and structure and Ω_1, Ω_2 be convex subsets of Ω . For $f \in \text{Lip}(\Omega) \cap L^1(\Omega)$, the following holds:*

$$\int_{\Omega_2} \int_{\Omega_1} |f(z) - f(y)| dz dy \leq 2^{n-1} \frac{\text{diam}(\Omega)}{n} (\mu(\Omega_1) + \mu(\Omega_2)) \int_{\Omega} |\nabla f(y)| dy.$$

Proof. The type of argument used here can be found in Aspects of Sobolev-Type Inequalities [28].

Let γ be a path from z to y . The definition of a gradient gives us:

$$|f(z) - f(y)| \leq \int_{\gamma} |\nabla f(s)| ds.$$

Note that if we are in a 1-dimensional space, a convex subset is a line. The desired inequality follows from expanding γ to Ω , and then noting that integrating over x

and y has the effect of multiplying the right hand side by

$$\mu(\Omega_1)\mu(\Omega_2) \leq \text{diam}(\Omega)(\mu(\Omega_1) + \mu(\Omega_2)).$$

Because z and y are in the same convex region Ω with an Euclidean distance, we can let the path γ be a straight line:

$$|f(z) - f(y)| \leq \int_0^{|y-z|} |\nabla f\left(z + \rho \frac{y-z}{|y-z|}\right)| d\rho.$$

We integrate this over $z \in \Omega_1, y \in \Omega_2$. To get a nice bound, we will use a trick from Korevaar and Schoen [23]. We split the path into two halves. For each half, we switch into and out of polar coordinates in a way that avoids integrating $\frac{1}{s}$ near $s = 0$. This allows us to have a bound which depends on the volumes of Ω_1 and Ω_2 rather than Ω .

First, we consider the half of the path which is closer to $y \in \Omega_2$. $I_\Omega(\cdot)$ is the indicator function for Ω .

$$\int_{\Omega_1} \int_{\Omega_2} \int_{\frac{|y-z|}{2}}^{|y-z|} |\nabla f(z + \rho \frac{y-z}{|y-z|})| I_\Omega(z + \rho \frac{y-z}{|y-z|}) d\rho dy dz.$$

We change of variable so that $y - z = s\theta$. That is, $|y - z| = s$ and $\frac{y-z}{|y-z|} = \theta$. Note that $\text{diam}(\Omega)$ is an upper bound on the distance between y and z .

$$\dots = \int_{\Omega_1} \int_{S^{n-1}} \int_0^{\text{diam}(\Omega)} \int_{s/2}^s |\nabla f(z + \rho\theta)| I_\Omega(z + \rho\theta) s^{n-1} d\rho ds d\theta dz.$$

We switch the order of integration. Now, ρ will be between 0 and $\text{diam}(\Omega)$ and s will be between ρ and $\min(2\rho, \text{diam}(\Omega))$. This allows us to integrate with respect to s .

$$\begin{aligned} \dots &= \int_{\Omega_1} \int_{S^{n-1}} \int_0^{\text{diam}(\Omega)} \int_{\rho}^{\min(2\rho, \text{diam}(\Omega))} |\nabla f(z + \rho\theta)| I_\Omega(z + \rho\theta) s^{n-1} ds d\rho d\theta dz \\ &= \int_{\Omega_1} \int_{S^{n-1}} \int_0^{\text{diam}(\Omega)} |\nabla f(z + \rho\theta)| I_\Omega(z + \rho\theta) \frac{(\min(2\rho, \text{diam}(\Omega)))^n - \rho^n}{n} d\rho d\theta dz. \end{aligned}$$

Now we reverse the change of variables to set $y = z + \rho\theta$. Since our integral includes an indicator function at $z + \rho\theta$, we have $y \in \Omega$.

$$\int_{\Omega_1} \int_{\Omega} |\nabla f(y)| \frac{(\min(2|y-z|, \text{diam}(\Omega)))^n - |y-z|^n}{n|y-z|^{n-1}} dy dz.$$

Let's consider the possible values of $\frac{(\min(2|y-z|, \text{diam}(\Omega)))^n - |y-z|^n}{n|y-z|^{n-1}}$.

If $|y-z| < \frac{\text{diam}(\Omega)}{2}$, then $\min(2|y-z|, \text{diam}(\Omega)) = 2|y-z|$. This gives us:

$$\begin{aligned} \frac{(\min(2|y-z|, \text{diam}(\Omega)))^n - |y-z|^n}{n|y-z|^{n-1}} &= \frac{2^n |y-z|^n - |y-z|^n}{n|y-z|^{n-1}} \\ &= \frac{2^n - 1}{n} |y-z| \\ &\leq \frac{\text{diam}(\Omega)(2^n - 1)}{2n}. \end{aligned}$$

Otherwise, if $|y-z| \geq \frac{\text{diam}(\Omega)}{2}$, then $\min(2|y-z|, \text{diam}(\Omega)) = \text{diam}(\Omega)$. This gives us:

$$\begin{aligned} \frac{(\min(2|y-z|, \text{diam}(\Omega)))^n - |y-z|^n}{n|y-z|^{n-1}} &= \frac{\text{diam}(\Omega)^n - |y-z|^n}{n|y-z|^{n-1}} \\ &\leq 2^{n-1} \frac{\text{diam}(\Omega)^n - |y-z|^n}{n \text{diam}(\Omega)^{n-1}} \\ &\leq 2^{n-1} \frac{\text{diam}(\Omega)^n}{n \text{diam}(\Omega)^{n-1}} \\ &= 2^{n-1} \frac{\text{diam}(\Omega)}{n}. \end{aligned}$$

Both cases are dominated by $2^{n-1} \frac{\text{diam}(\Omega)}{n}$. We place this into the original integral:

$$\begin{aligned} \int_{\Omega_1} \int_{\Omega} |\nabla f(y)| \frac{(\min(2|y-z|, \text{diam}(\Omega)))^n - |y-z|^n}{n|y-z|^{n-1}} dy dz \\ \leq \int_{\Omega_1} \int_{\Omega} |\nabla f(y)| 2^{n-1} \frac{\text{diam}(\Omega)}{n} dy dz \\ = 2^{n-1} \frac{\text{diam}(\Omega)}{n} \mu(\Omega_1) \int_{\Omega} |\nabla f(y)| dy. \end{aligned}$$

This is an upper bound for

$$\int_{\Omega_1} \int_{\Omega_2} \int_{\frac{|y-z|}{2}}^{|y-z|} |\nabla f(z + \rho \frac{y-z}{|y-z|})| I_{\Omega}(z + \rho \frac{y-z}{|y-z|}) d\rho dy dz.$$

We can apply the same argument to the half of the geodesic closest to $z \in \Omega_1$, after first substituting $\rho' = |y - z| - \rho$:

$$\begin{aligned} & \int_{\Omega_1} \int_{\Omega_2} \int_0^{\frac{|y-z|}{2}} |\nabla f(z + \rho \frac{y-z}{|y-z|})| I_{\Omega}(z + \rho \frac{y-z}{|y-z|}) d\rho dy dz \\ &= \int_{\Omega_2} \int_{\Omega_1} \int_{\frac{|y-z|}{2}}^{|y-z|} |\nabla f(y + \rho' \frac{z-y}{|z-y|})| I_{\Omega}(y + \rho' \frac{z-y}{|z-y|}) d\rho dz dy \\ &\leq 2^{n-1} \frac{\text{diam}(\Omega)}{n} \mu(\Omega_2) \int_{\Omega} |\nabla f(y)| dy. \end{aligned}$$

Combining these with the original inequality, we have

$$\int_{\Omega_2} \int_{\Omega_1} |f(z) - f(y)| dz dy \leq 2^{n-1} \frac{\text{diam}(\Omega)}{n} (\mu(\Omega_1) + \mu(\Omega_2)) \int_{\Omega} |\nabla f(y)| dy.$$

□

Notation 2.1.2. Let X be an admissible Euclidean polytopal complex of dimension n .

Definition 2.1.3. Let B be a ball of radius r whose center is on a D -dimensional face with the property that B intersects no other D -dimensional faces. We define wedges W_k of B to be the closures of each of the connected components of $B - X^{(n-1)}$.

Note that for any point z in X , a ball $B(z, r)$ satisfying the above criteria exists: for each dimension D , we can take any point $z \in X^D - X^{(D-1)}$ and any $r < d(z, X^{(D-1)})$ and create $B = B(z, r) \subset X$. Then B is a ball of radius r whose center is on a D -dimensional face, and B intersects no other D -dimensional faces. In essence, the wedges, W_k , are formed when the $(n-1)$ skeleton slices the ball B into pieces. This construction tells us that each W_k has diameter at most $2r$, as each of the points in W_k is within distance r of z , and z is included in W_k .

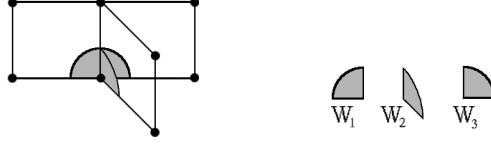


Figure 2.1: Complex with shaded ball B (left); the three wedges for B (right).

Example 2.1.4. In figure 2.1.4 we have an example of a 2 dimensional complex with a shaded ball centered at a vertex. This ball has three wedges; one for each of the two dimensional faces that share the vertex. Note that each wedge is a fraction of a sphere.

Definition 2.1.5. We say that X has degree bounded by M if M is the maximal number of edges in $X^{(1)}$ that can share a vertex in $X^{(0)}$.

Definition 2.1.6. We say that X has edge lengths bounded below by ℓ if

$$0 < \ell \leq \inf_{v,w \in X^{(0)}} d(v, w).$$

Note that having degree bounded by M implies that the maximum number of k dimensional faces that can share a lower dimensional face is also M . This tells us that sufficiently small balls will be split into at most M wedges. Note that if X has degree bounded by M , then $X^{(k)}$ will as well. When X has degree bounded by M , volume doubling will occur locally with a uniform constant. In particular, when the edge lengths are bounded below by ℓ the strong statement:

$$\mu(B(x, cr)) \leq Mc^N \mu(B(x, r))$$

will hold whenever $cr \leq \ell$. For balls in X , N will equal n , the dimension of X . If we restrict to balls in $X^{(k)}$, then this holds with $N = k$.

To show a local Poincaré inequality on X , we will split the balls, which are not necessarily convex, up into smaller overlapping pieces which are. We will do this

using the wedges. We can use a chaining argument in order to move through B from one of the W_k to another. Note that this uses the fact that our space X is admissible. We will say W_k and W_j are adjacent if they share an $n - 1$ dimensional face, and let $N(j)$ be the list of indices of faces adjacent to W_j including j . In order to create paths which we can integrate over, we need an overlapping region between the adjacent faces. For $k \in N(j)$, let $W_{k,j} = W_{j,k}$ be the largest subset of $W_k \cup W_j$ which has the property that $W_k \cup W_{k,j}$ and $W_j \cup W_{k,j}$ are both convex. Then, for each x in $W_{k,j}$ there is a way of describing the rays between x and W_k in a distance preserving manner as one would have in R^n . This will justify our use of the ρ in the calculation below.

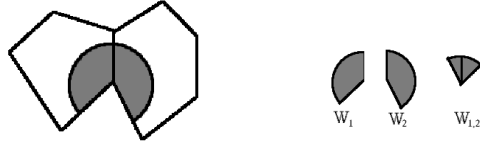


Figure 2.2: Complex with shaded ball B (left); the two wedges for B and a region which overlaps both of them(right).

Example 2.1.7. In figure 2.1.7 we have a complex and ball with two adjacent wedges. The union of the wedges, W_1 and W_2 , is not convex, so we form the region $W_{1,2}$. In this example, both $W_1 \cup W_{1,2}$ and $W_2 \cup W_{1,2}$ are half circles.

Theorem 2.1.8. *Let X be an admissible Euclidean polytopal complex of dimension n with degree bounded by M . For each $z \in X$ there exists $r > 0$ so that for $B = B(z, r)$ and its corresponding wedges, $W_{i,j}$, the following holds for $f \in \text{Lip}(X) \cap L^1(B)$:*

$$\|f - f_B\|_{1,B} \leq 2M \max_{k,j \in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2 \right) \frac{2^n r (\mu(W_k) + \mu(W_{j,k}))}{n \mu(W_{j,k})} \|\nabla f\|_{1,B}.$$

Proof. For a given $z \in X$ let D be the dimension such that $z \in X^{(D)} - X^{(D-1)}$. Pick $r < d(z, X^{(D-1)})$. Let $B = B(z, r)$. For x in B we have:

$$\begin{aligned} |f(x) - f_B| &= \left| \int_B \frac{1}{\mu(B)} f(x) dy - \int_B \frac{1}{\mu(B)} f(y) dy \right| \\ &\leq \frac{1}{\mu(B)} \int_B |f(x) - f(y)| dy. \end{aligned}$$

We would like to apply Lemma 2.1.1 to this; however, B is not necessarily convex. We will construct a path from x to y using a finite number of straight lines, where each of the line segments is contained in a convex region. For simplicity, we will consider $x \in W_i$ and $y \in W_k$. It is quite possible that these two wedges are not contained in a convex subset of B . We need to use the fact that our space is locally $(n-1)$ -chainable by looking at a chain in $B - \{z\}$ starting at W_i and ending at W_k . The pieces of the chain will move us from a point in W_j into a connecting point in $W_{j,l}$, and then from that connecting point in $W_{j,l}$ into a point in W_l .

Formulated more precisely, there is a sequence of indices, $\sigma(1) = i, \dots, \sigma(l) = k$ that corresponds to this chain of W 's, so that for each j , $W_{\sigma(j)}$ and $W_{\sigma(j+1)}$ are adjacent, and none of the indices repeat. We can take points in these regions; $z_1 \in W_{\sigma(1)}$, $z_2 \in W_{\sigma(1), \sigma(2)}$, \dots , $z_{2j-1} \in W_{\sigma(j)}$ and $z_{2j} \in W_{\sigma(j), \sigma(j+1)}$. Note that each pair in this sequence is located in a convex region— either $W_{\sigma(j)} \cup W_{\sigma(j), \sigma(j+1)}$ or $W_{\sigma(j+1)} \cup W_{\sigma(j), \sigma(j+1)}$. The line segments between these points will define our path γ from x to y .

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(z_1) + f(z_1) - \dots + f(z_{2l}) - f(y)| \\ &\leq |f(x) - f(z_1)| + |f(z_1) - f(z_2)| + \dots + |f(z_{2l}) - f(y)| \\ &= |f(x) - f(z_1)| + \sum_{j=1}^{l-1} (|f(z_{2j}) - f(z_{2j-1})| + |f(z_{2j}) - f(z_{2j+1})|) \\ &\quad + |f(z_{2l}) - f(y)|. \end{aligned}$$

Since it didn't matter which z 's we chose, as long as they were in the proper sets, we can average the pieces over all of the possible z 's.

$$\begin{aligned}
|f(x) - f(y)| &\leq \oint_{W_{i,\sigma(1)}} |f(x) - f(z_1)| dz_1 \\
&\quad + \sum_{j=1}^{l-1} \left(\oint_{W_{\sigma(j)}} \oint_{W_{\sigma(j),\sigma(j+1)}} |f(z_{2j}) - f(z_{2j-1})| dz_{2j} dz_{2j-1} \right. \\
&\quad \left. + \oint_{W_{\sigma(j+1)}} \oint_{W_{\sigma(j),\sigma(j+1)}} |f(z_{2j}) - f(z_{2j+1})| dz_{2j} dz_{2j+1} \right) \\
&\quad + \oint_{W_{\sigma(l),k}} |f(z_{2l}) - f(y)| dz_{2l}.
\end{aligned}$$

We will not want to keep track of the exact path between every pair of regions, although in specific examples one may want to do that in order to achieve a tighter bound. Rather, it is useful simply integrate over all pairs of neighboring wedges, as this will include everything in our path.

$$\begin{aligned}
|f(x) - f(y)| &\leq \sum_{l \in N(i)} \oint_{W_{i,l}} |f(x) - f(z)| dz \\
&\quad + \sum_j \sum_{l \neq i, l \in N(j)} \oint_{W_i} \oint_{W_{j,l}} |f(z) - f(w)| dz dw \\
&\quad + \sum_{j \in N(k)} \oint_{W_{j,k}} |f(z) - f(y)| dz.
\end{aligned}$$

This new inequality will hold for x and y in any pair of W_i and W_k with $k \neq i$. If we expand our notation so that $W_{i,i} = W_i$, then this will hold when x and y are in the same set $W_k = W_i$. To integrate over all $y \in B$, we can split the integral into two parts; one where x and y are both in W_i , and then add it to the second where y is in one of the $W_k \neq W_i$. Similarly, we can integrate over x in W_i and

then sum over i .

$$\begin{aligned}
& \frac{1}{\mu(B)} \int_B \int_B |f(x) - f(y)| dy dx \\
& \leq \frac{1}{\mu(B)} \left(\sum_{i,k} \sum_{l \in N(i)} \int_{W_i} \int_{W_k} \int_{W_{i,l}} |f(x) - f(z)| dz dy dx \right. \\
& \quad + \sum_{i,k,j} \sum_{l \in N(j)} \int_{W_i} \int_{W_k} \int_{W_{j,l}} \int_{W_l} |f(z) - f(w)| dw dz dy dx \\
& \quad \left. + \sum_{i,k} \sum_{j \in N(k)} \int_{W_i} \int_{W_k} \int_{W_{j,k}} |f(z) - f(y)| dz dy dx \right) \\
& = \sum_i \sum_{l \in N(i)} \int_{W_i} \int_{W_{i,l}} |f(x) - f(z)| dz dx \\
& \quad + \mu(B) \sum_j \sum_{l \in N(j)} \int_{W_{j,l}} \int_{W_l} |f(z) - f(w)| dw dz \\
& \quad + \sum_k \sum_{j \in N(k)} \int_{W_k} \int_{W_{j,k}} |f(z) - f(y)| dz dy.
\end{aligned}$$

We can combine this into one double sum by setting $x = w$ and $y = w$ as well as reindexing so that $i = j$ and $l = k$.

$$\ldots \leq \sum_k \sum_{j \in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2 \right) \int_{W_k} \int_{W_{j,k}} |f(z) - f(w)| dz dw.$$

Applying lemma 2.1.1 with $\Omega = W_k \cup W_{j,k}$, $\Omega_1 = W_{j,k}$, $\Omega_2 = W_k$, and $\text{diam}(\Omega) \leq 2r$ to each of the pieces we find:

$$\ldots \leq \sum_k \sum_{j \in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2 \right) \frac{2^n r (\mu(W_k) + \mu(W_{j,k}))}{n \mu(W_{j,k})} \int_{W_k \cup W_{j,k}} |\nabla f(y)| dy.$$

Note that points in the sets $W_k \cup W_{j,k}$ are counted at most $2M$ times, since each of the W_k has at most M neighbors. This allows us to combine the sums to find:

$$\begin{aligned}
& \frac{1}{\mu(B)} \int_B \int_B |f(x) - f(y)| dy dx \\
& \leq 2M \max_{k,j \in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2 \right) \frac{2^n r (\mu(W_k) + \mu(W_{j,k}))}{n \mu(W_{j,k})} \int_B |\nabla f(y)| dy.
\end{aligned}$$

This is the desired result. \square

Now that we have the inequality when $p = 1$, we can use a trick to extend it to other values of p .

Lemma 2.1.9. *If for any $f \in \text{Lip}(X)$ we have:*

$$\|f - f_B\|_{1,B} \leq Cr \|\nabla f\|_{1,B}$$

for $B = B(z, r)$ then

$$\begin{aligned} \inf_{c \in (-\infty, \infty)} \|f - c\|_{p,B} &\leq pCr \|\nabla f\|_{p,B} \text{ and} \\ \|f - f_B\|_{p,B} &\leq 2pCr \|\nabla f\|_{p,B}. \end{aligned}$$

holds for $1 \leq p < \infty$.

Proof. Let $g(x) = |f(x) - c_f|^p \text{sign}(f(x) - c_f)$. Note that g is in $\text{Lip}(X)$. Then if ∇f is the gradient of f , we have that $p|f(x) - c_f|^{p-1}|\nabla f(x)|$ is the length of the gradient of g .

Pick a value of c_f so that $g_B = \int_B g(x) dx = 0$. (One will exist; we consider g_B as a function of c_f and apply the intermediate value theorem.)

Applying our assumption to g , we have:

$$\begin{aligned} \int_B |g(x) - 0| dx &\leq C \int_B |\nabla g(x)| dx \\ &= C \int_B p|f(x) - c_f|^{p-1} |\nabla f(x)| dx. \end{aligned}$$

Now we use Hölder to find:

$$\int_B |f(x) - c_f|^{p-1} |\nabla f(x)| dx \leq \left(\int_B (|f(x) - c_f|^{p-1})^q dx \right)^{1/q} \left(\int_B |\nabla f(x)|^p dx \right)^{1/p}.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $(p-1)q = p$. Combining this with the above inequality gives us:

$$\left(\int_B |f(x) - c_f|^p dx \right)^{1/p} \leq pCr \left(\int_B |\nabla f(x)|^p dx \right)^{1/p}.$$

When we take an infimum, we find that

$$\inf_c \left(\int_B |f(x) - c|^p dx \right)^{1/p} \leq \left(\int_B |f(x) - c_f|^p dx \right)^{1/p}.$$

When we combine these, we have

$$\inf_c \|f - c\|_{p,B} \leq pCr \|\nabla f\|_{p,B}.$$

In the case where $p = 2$, it is easy to compute the infimum exactly. Consider

$$\begin{aligned} h(c) &= \int_B |f(x) - c|^2 dx \\ &= \int_B f(x)^2 dx - 2c \int_B f(x) dx + c^2 \mu(B). \end{aligned}$$

This is a parabola whose minimum occurs at $c = \frac{1}{\mu(B)} \int_B f(x) dx = f_B$. Its minimum is the same as that of $\sqrt{h(c)}$, and so this gives us

$$\|f - f_B\|_{2,B} \leq 2Cr \|\nabla f\|_{2,B}.$$

When $p \neq 2$, we can use Jensen's inequality to get the average. We do this by noticing:

$$\begin{aligned} \int_B |f_B - c|^p dx &= \left| \int_B (f - c) dx \right|^p \\ &\leq \int_B |f - c|^p dx. \end{aligned}$$

This tells us that

$$\begin{aligned} \|f - f_B\|_{p,B} &\leq \inf_c \|f - c\|_{p,B} + \|f_B - c\|_{p,B} \\ &\leq 2 \inf_c \|f - c\|_{p,B} \\ &\leq 2pCr \|\nabla f\|_{p,B}. \end{aligned}$$

□

Definition 2.1.10. We say that X has solid angle bound α if for each $z \in X^{(D)} - X^{(D-1)}$ and $r < d(z, X^{(D-1)})$ the wedges of the ball $B(z, r)$ satisfy

$$\alpha \leq \frac{\mu(W_k)}{\mu(r^n S^{(n-1)})} \leq 1.$$

Note that the right hand side of the inequality reflects the fact that each of the W_k is a subset of an Euclidean ball.

If we have a uniform bound on the solid angles formed, then the constant in Theorem 2.1.8 will simplify.

Corollary 2.1.11. *Suppose X is an admissible n -dimensional Euclidean polytopal complex with solid angle bound α , and $f \in \text{Lip}(X)$. For each $z \in X$ there exists $r > 0$ so that for $B = B(z, r)$ we have*

$$\|f - f_B\|_{p,B} \leq C_X p r \|\nabla f\|_{p,B}$$

where the constant $C_X = \frac{2^{3n+3}M^2}{\alpha n}$ depends only on the space X .

Proof. We need to bound $\max_{k,j \in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2 \right) \frac{\mu(W_k) + \mu(W_{j,k})}{\mu(W_{j,k})}$ from Theorem 2.1.8. Since we will want to bound the $\mu(W_{j,k})$, we need a way to compare its size to the volume of the other W_j . We can subdivide the space initially by cutting each piece in half in each of the n dimensions, so that there are at most $M' = 2^n M$ pieces. When the W'_k and W'_j are adjacent, this tells us that $W'_{j,k}$ has a volume which is larger than $\min(\mu(W'_j), \mu(W'_k))$. Thus $\frac{\mu(W_k) + \mu(W_{j,k})}{\mu(W_{j,k})} \leq 2$.

To bound $\frac{\mu(B)}{\mu(W_k)}$ we will need the solid angle bound. Combining the solid angle bound with the factor of 2^{-n} decrease in wedge size gives us the modified inequality:

$$\mu(W'_k) \leq 2^{-n} \mu(r^n S^{(n-1)}) \leq \frac{\mu(W'_k)}{\alpha}.$$

Summing the left hand side of the inequality over k tells us that

$$\mu(B) \leq M 2^n 2^{-n} \mu(r^n S^{(n-1)}).$$

If we multiply the right hand side of the inequality by $M2^n$, we have

$$M\mu(r^n S^{(n-1)}) \leq \frac{M2^n \mu(W'_k)}{\alpha}.$$

Combining these two inequalities, we find that:

$$\frac{\mu(B)}{\mu(W'_k)} \leq \frac{M2^n}{\alpha}.$$

We can substitute these into our constant to get:

$$2M' \max_{k,j \in N(k)} \left(\frac{\mu(B)}{\mu(W'_k)} + 2 \right) \frac{2^n r (\mu(W'_k) + \mu(W'_{j,k}))}{n \mu(W'_{j,k})} \leq 2M2^n \left(\frac{M2^n}{\alpha} + 2 \right) \frac{2^{n+1} r}{n}.$$

This combined with theorem 2.1.8 and lemma 2.1.9 gives us that:

$$\|f - f_B\|_{p,B} \leq \frac{2^{3n+3} M^2}{\alpha n} p r \|\nabla f\|_{p,B}.$$

□

We would like to extend these theorems so that the radius is not dependent on the center of the ball. To do so, we will first show a weaker Poincaré inequality, and then we will extend it via a Whitney type covering to a stronger version.

Theorem 2.1.12. *Suppose X is an admissible n -dimensional Euclidean polytopal complex with solid angle bound α and edge lengths bounded below by ℓ , and*

$f \in \text{Lip}(X)$. When $\kappa = 6 \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1 \right)^n$, the following inequality holds for each $z \in X$ and each $0 \leq r \leq R_0$ where $R_0 := \frac{\inf_{v,w \in X^0} d(v,w)}{6 \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1 \right)^n} = \frac{\ell}{\kappa}$.

$$\int_{B(z,r)} |f(x) - f_{B(z,r)}| dx \leq r C_{Weak} \int_{B(z,\kappa r)} |\nabla f(x)| dx$$

where $C_{Weak} = \frac{2^{3n+3} M^3 \kappa^{N+1}}{\alpha n}$.

Proof. Let $z \in X$ and $r \leq \frac{\inf_{v,w \in X^0} d(v,w)}{6 \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1 \right)^n}$ be given. If $d(z, X^{(n-1)}) > r$, then the result follows as a weaker version of Corollary 2.1.11. Otherwise, we will need to

find a point v_k which has the property that it is on a k -skeleton, and there are no other faces in the k -skeleton that are intersected by $B(v, d(v, z) + r)$. We will do this by descending down the skeletons.

If there is a point within r of z with this property, we will use it.

If not, set $r_0 = 3r$. Then there is a k such that the lowest dimensional skeleton that is intersected by $B(z, r_0)$ is $X^{(k)}$, and $X^{(k)}$ is intersected by $B(z, r_0)$ at at least two points v_k and w_k on two different faces. If these faces did not intersect, then they would be at least distance $\inf_{v, w \in X^{(0)}} d(v, w)$ from one another. This would imply that $\inf_{v, w \in X^{(0)}} d(v, w) \leq 2r_0 = 6r$, a contradiction. Thus those two faces intersect in a smaller j -dimensional face. Call v_j the point on the j -dimensional face which minimizes $\min(d(v_j, v_k), d(v_j, w_k))$. These three points form a triangle with angle $v_k v_j w_k \geq \alpha$, where α is the smallest interior angle in X . Note that this angle is bounded by the assumption $\alpha \leq \frac{\mu(W_k)}{\mu(r^n S^{n-1})}$. The triangle that would maximize the minimum distance to this new point, $\min(d(v_j, v_k), d(v_j, w_k))$ is an isosceles one with angle $v_k v_j w_k = \alpha$. The law of cosines tells us that for the isosceles triangle, $d(v_k, w_k)^2 = 2d(v_k, v_j)^2(1 - \cos(\alpha))$, and so for a general triangle, $d(v_k, v_j) \leq \frac{d(v_k, w_k)}{\sqrt{2(1 - \cos(\alpha))}} \leq \frac{2r_0}{\sqrt{2(1 - \cos(\alpha))}}$.

If this v_j works, we're done. Otherwise, we will have at least two points, v_j and w_j within $\left(\frac{2}{\sqrt{2(1 - \cos(\alpha))}} + 1\right) r_0$ of z . We will repeat the process by taking new r 's of the form $r_{i+1} = \left(\frac{2}{\sqrt{2(1 - \cos(\alpha))}} + 1\right) r_i$ until we have a point which works. Note that each time we repeat it, we find a point on a lower dimensional skeleton. The worst case scenario will have us repeat this n times until we're left with at least one point on $X^{(0)}$. The largest radius that we could require is $R = \left(\frac{2}{\sqrt{2(1 - \cos(\alpha))}} + 1\right)^n 3r$. Using this R , we can show that $B(v, R)$ does not intersect two vertices. The con-

dition $r \leq \frac{\inf_{v,w \in X^0} d(v,w)}{6 \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1 \right)^n}$ tells us that $R \leq \frac{1}{2} \inf_{v,w \in X^0} d(v,w)$. As two vertices cannot be closer than the closest pair, $B(v, R)$ contains at most one vertex.

This construction gives us a center, v , on a k -dimensional face, and a radius, $R \leq \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1 \right)^n 3r$ so that $B(v, R)$ intersects only the k -dimensional face that v is on. This allows us first to recenter our ball around v and then to apply Corollary 2.1.11 to f on $B(v, R)$. Then, as $\kappa = 6 \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1 \right)^n$, we find $B(v, R) \subset B(z, \kappa r)$.

$$\begin{aligned}
\int_{B(z,r)} |f(x) - f_{B(z,r)}| dx &= \frac{1}{\mu(B(z,r))} \int_{B(z,r)} \int_{B(z,r)} |f(x) - f(y)| dx dy \\
&\leq \frac{1}{\mu(B(z,r))} \int_{B(v,R)} \int_{B(v,R)} |f(x) - f(y)| dx dy \\
&\leq \frac{\mu(B(v,R))}{\mu(B(z,r))} \int_{B(v,R)} |f(x) - f_{B(v,R)}| dx \\
&\leq \frac{\mu(B(v,R))}{\mu(B(z,r))} \frac{2^{3n+3} M^2}{\alpha n} R \int_{B(v,R)} |\nabla f(x)| dx \\
&\leq \frac{\mu(B(z, \kappa r))}{\mu(B(z,r))} \frac{2^{3n+3} M^2}{\alpha n} \kappa r \int_{B(z, \kappa r)} |\nabla f(x)| dx \\
&\leq M \kappa^N \frac{2^{3n+3} M^2}{\alpha n} \kappa r \int_{B(z, \kappa r)} |\nabla f(x)| dx.
\end{aligned}$$

□

2.2 Whitney Covers

We would like to strengthen the weak version of the Poincaré inequality. We'll do this by using a Whitney type covering of the ball, $E = B(z, r)$. Once we have this cover, we can use a chaining argument to allow us to replace

$$\kappa = 6 \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1 \right)^n \text{ with } 1.$$

Given our set we will consider a collection F of balls such that

- (1) $B \in \mathcal{F}$ are disjoint.

(2) If we expand the balls to ones with twice the radius, we cover all of E .

$$\cup_{B \in \mathcal{F}} 2B = E.$$

(3) For any $B \in \mathcal{F}$, its radius is $r_B = 10^{-3}\kappa^{-1}d(B, \partial E)$. This implies $10^3\kappa B \subset E$.

Note that this also tells us that the distance from the center of B to the boundary of E is $(10^{-3}\kappa^{-1} + 1)d(B, \partial E) = (10^3\kappa + 1)r_B$.

(4) $\sup_{x \in E} |\{B \in \mathcal{F} | x \in 36\kappa B\}| \leq K$.

Note that the constant κ depends on X but not on the specific choice of E . We will show in the following lemma that K is independent of E as well.

Lemma 2.2.1. *Property 4 is satisfied for $K \leq C_{vol}^{\log_2(8(10^3\kappa+1))}$. When E is a ball in X which intersects only one vertex, we have $K \leq M(8(1 + 10^3\kappa))^N$.*

Proof. Let a point $x \in E$ be given. Let $B(y, r_y) \in \mathcal{F}$ be a ball centered at y with the property that $x \in 36\kappa B(y, r_y)$. Then:

$$\begin{aligned} d(x, y) &\leq 36\kappa r_y = 36\kappa 10^{-3}\kappa^{-1}d(y, \partial E) \\ &\leq .036(d(x, y) + d(x, \partial E)). \end{aligned}$$

When we solve for $d(x, y)$, we have:

$$d(x, y) \leq \frac{.036}{1 - .036}d(x, \partial E).$$

The triangle inequality tells us that

$$d(x, \partial E) - d(x, y) \leq d(y, \partial E) \leq d(y, x) + d(x, \partial E).$$

We use this to bound $d(y, \partial E)$.

$$\frac{1 - .072}{1 - .036}d(x, \partial E) \leq d(y, \partial E) \leq \frac{1}{1 - .036}d(x, \partial E).$$

Because $r_y = (10^3\kappa + 1)^{-1}d(y, \partial E)$, this tells us that the radius r_y is bounded by

$$(1 + 10^3\kappa)^{-1} \frac{1 - .072}{1 - .036} d(x, \partial E) \leq r_y \leq \frac{(1 + 10^3\kappa)^{-1}}{1 - .036} d(x, \partial E).$$

These inequalities hold for any $B(y, r_y) \in \mathcal{F}$ with $x \in 36\kappa B(y, r_y)$. Each of the balls, B_y will be contained in $B_x := B(x, r_1 d(x, \partial E))$ where $r_1 = \frac{(1+10^3\kappa)^{-1} + .036}{1 - .036}$. The B_y have radius at least $r_2 d(x, \partial E)$ where $r_2 = (1 + 10^3\kappa)^{-1} \frac{1 - .072}{1 - .036}$. We also know that the B_y are disjoint. This tells us that:

$$|\{B \in \mathcal{F} | x \in 36\kappa B\}| \min_{B_y \in \mathcal{F} \cap B_x} \mu(B_y) \leq \sum_{B_y \in \mathcal{F} \cap B_x} \mu(B_y) \leq \mu(B_x).$$

We can use volume doubling to compare the size of $\min_{B_y \in \mathcal{F} \cap B_x} \mu(B_y)$ and $\mu(B_x)$.

Note that $r_1 < 2$ and $\frac{1}{2(1+10^3\kappa)} < r_2$. This tells us that

$$B(x, r_1 d(x, \partial E)) \subset B(y, 8(1 + 10^3\kappa) r_2 d(x, \partial E)).$$

$$\mu(B_x) \leq C_{vol}^{\log_2(8(1+10^3\kappa))} \mu(B_y).$$

Combining these inequalities and taking the supremum over $x \in E$ gives us:

$$\sup_{x \in E} |\{B \in \mathcal{F} | x \in 36\kappa B\}| \leq C_{vol}^{\log_2(8(1+10^3\kappa))}.$$

Note that this only depends on κ and C_{vol} . When E intersects only one vertex we have:

$$\mu(B_x) \leq M 2^{2N} (2(1 + 10^3\kappa))^N \mu(B_y).$$

This gives us a more refined estimate. Here N is the dimension of X :

$$\sup_{x \in E} |\{B \in \mathcal{F} | x \in 36\kappa B\}| \leq M (8(1 + 10^3\kappa))^N.$$

□

We will first describe properties of this collection, and then we will use them to show a Poincaré inequality. This is a modified version of the argument found in [28]. We can also use this technique to take a Poincaré inequality on a small ball and extend it to one on a larger ball whenever we have volume doubling. This increases the constant involved, so it cannot be done indefinitely, but given a fixed radius we will be able to have inequalities that hold up to balls of that size.

We begin with a bit of notation. Let $B_z \in \mathcal{F}$ be a ball such that $z \in 2B_z$. Note that there may be more than one; we will pick one arbitrarily. As z is the center of E , we will call B_z the central ball. For a ball, B , call the center x_B , and fix γ_B , a distance minimizing curve from z to x_B .

Lemma 2.2.2. *For any $B \in \mathcal{F}$ we have*

$$d(\gamma_B, \partial E) \geq \frac{1}{2}d(B, \partial E) = \frac{1}{2}\kappa 10^3 r_B.$$

If $B' \in \mathcal{F}$ has the property $2B' \cap \gamma_B \neq \emptyset$, then $r_{B'} \geq \frac{1}{4}r_B$.

Proof. This first claim will follow from multiple applications of the triangle inequality. Let α be the point in γ_B which is closest to the boundary:

$d(\gamma_B, \partial E) = d(\alpha, \partial E)$. Then we can bound r_E :

$$d(z, \alpha) + d(\alpha, \partial E) \geq d(z, \partial E) = r_E$$

and we can bound the distance from B to ∂E :

$$d(x_B, \alpha) + d(\alpha, \partial E) \geq d(x_B, \partial E) \geq d(B, \partial E).$$

Summing them, we find that:

$$d(z, \alpha) + d(x_B, \alpha) + 2d(\alpha, \partial E) \geq r_E + d(B, \partial E).$$

As z is on γ_B and α minimizes the distance to the boundary, we have

$d(z, \alpha) + d(\alpha, x_B) = d(z, x_B) \leq r_E$. Putting this into the inequality, we find:

$$\begin{aligned} r_E + 2d(\alpha, \partial E) &\geq r_E + d(B, \partial E) \\ d(\alpha, \partial E) &\geq \frac{1}{2}d(B, \partial E). \end{aligned}$$

The second part follows from the fact:

$$\frac{1}{2}d(B, \partial E) \leq d(\gamma_B, \partial E) \leq d(\gamma_B \cap 2B', \partial E).$$

If α' is the point in $\gamma_B \cap 2\bar{B}'$ and β' is the point in \bar{B}' that realizes the distance to the boundary ∂E , then we have

$$\begin{aligned} d(\gamma_B \cap 2B', \partial E) = d(\alpha', \partial E) &\leq d(\alpha', x_{B'}) + d(x_{B'}, \beta') + d(B', \partial E) \\ &\leq 2r_{B'} + r_{B'} + d(B', \partial E). \end{aligned}$$

Combining this with fact (3), we have:

$$\begin{aligned} \frac{1}{2}10^3\kappa r_B &\leq 3r_{B'} + 10^3\kappa r_{B'} \\ \frac{1}{4}r_B &\leq \frac{10^3\kappa}{2(3 + 10^3\kappa)}r_B \leq r_{B'}. \end{aligned}$$

□

For each ball B in \mathcal{F} , we would like to define a string of balls, $\mathcal{F}(B)$, that takes B_z to B . Set $B_0 = B_z$. Then, for the first point on γ_B that is not contained in $2B_i$, take a ball B_{i+1} in F such that that point is contained in $2B_{i+1}$. As B_{i+1} is open, this guarantees that $2B_i \cap 2B_{i+1} \neq \emptyset$. We continue in this manner until $2B_{\ell-1} \cap 2B \neq \emptyset$. Then we set $B_\ell = B$. We label $\mathcal{F}(B) = \cup_{i=0}^{\ell} B_i$. Note that due to volume doubling, the chain will be finite. This chain will allow us to move from the central ball to any other ball in the cover of E . It is useful because neighboring balls are of comparable radii and volume.

Lemma 2.2.3. *For any $B \in \mathcal{F}$ and any $B_i, B_{i+1} \in \mathcal{F}(B)$ we can compare the radii where $r_j = r_{B_j}$ in the following manner:*

$$(1 + 10^{-2}\kappa^{-1})^{-1}r_i \leq r_{i+1} \leq (1 + 10^{-2}\kappa^{-1})r_i.$$

We also have $B_{i+1} \subset 6B_i$ and $B_i \subset 6B_{i+1}$, and so

$$\mu(6B_i \cap 6B_{i+1}) \geq \max\{\mu(B_i), \mu(B_{i+1})\}.$$

Proof. Let x_i and x_{i+1} be the centers of B_i and B_{i+1} respectively. By our construction, $2B_i \cap 2B_{i+1} \neq \emptyset$. This tells us that $d(x_i, x_{i+1}) \leq 2r_i + 2r_{i+1}$.

$$\begin{aligned} d(x_{i+1}, \partial E) &\leq d(x_{i+1}, x_i) + d(x_i, \partial E) \\ r_{i+1} + d(B_{i+1}, \partial E) &\leq (2r_i + 2r_{i+1}) + (r_i + d(B_i, \partial E)) \\ r_{i+1} + 10^3\kappa r_{i+1} &\leq 2r_{i+1} + 3r_i + 10^3\kappa r_i \\ r_{i+1} &\leq \frac{3 + 10^3\kappa}{10^3\kappa - 1}r_i = \left(1 + \frac{4}{10^3\kappa - 1}\right)r_i. \end{aligned}$$

This tells us that $r_{i+1} \leq (1 + 10^{-2}\kappa^{-1})r_i$. By a symmetric argument, we also get the lower bound.

To show set inclusions, we use the fact that

$$d(x_i, x_{i+1}) \leq 2r_i + 2r_{i+1} \leq 2r_i + 2(1 + 10^{-2}\kappa^{-1})r_i.$$

Since any point in B_{i+1} is within distance r_{i+1} of x_{i+1} , the triangle inequality tells us that it is within distance $2r_i + 2(1 + 10^{-2}\kappa^{-1})r_i + r_{i+1} \leq 6r_i$ of x_i . This gives us the inclusion $B_{i+1} \subset 6B_i$. The reverse holds by a symmetric argument, and so $\mu(6B_i \cap 6B_{i+1}) \geq \max\{\mu(B_i), \mu(B_{i+1})\}$ follows. \square

Lemma 2.2.4. *Given $B \in \mathcal{F}$ and $A \in \mathcal{F}(B)$ we have $B \subset (10^3\kappa + 9)A$.*

Proof. Let x_B and x_A be the centers of B and A respectively, and let α be a point in $2A \cap \gamma_B$. By 2.2.2, we know that $4r_A \geq r_B$. Note that α occurs on the distance minimizing curve γ_B between x_B which is the center of B and z , the center of the large ball E . This tells us that $d(z, x_B) = d(z, \alpha) + d(\alpha, x_B)$. Then

$$\begin{aligned}
d(x_A, x_B) &\leq d(x_A, \alpha) + d(\alpha, x_B) \\
&= d(x_A, \alpha) + (d(z, x_B) - d(z, \alpha)) \\
&\leq 2r_A + d(z, \partial E) - d(z, \alpha) \\
&\leq 2r_A + (d(z, \alpha) + d(\alpha, \partial E)) - d(z, \alpha) \\
&\leq 2r_A + d(\alpha, x_A) + d(x_A, \partial E) \\
&\leq 4r_A + (r_A + 10^3 \kappa r_A) = (10^3 \kappa + 5)r_A.
\end{aligned}$$

Because all points in B are within $r_B \leq 4r_A$ of x_B , they will be within $(10^3 \kappa + 5)r_A + 4r_A = (10^3 \kappa + 9)r_A$ of x_A . Thus, $B \subset (10^3 \kappa + 9)A$ holds. \square

We now have a number of lemmas that describe the geometry of the covering. We can use these to get an extension of our Poincaré inequality. We will do this by using this chain of balls to get a chain of inequalities. Our first step is to compare the average of neighboring balls in the chain.

Lemma 2.2.5. *For B_i and B_{i+1} neighboring balls in a chain $\mathcal{F}(B)$, we have*

$$|f_{6B_i} - f_{6B_{i+1}}| \leq C_{Weak} 18\kappa \frac{r_i}{\mu(B_i)} \int_{36\kappa B_i} |\nabla f(x)| d\mu(x)$$

whenever f satisfies

$$\|f - f_{6B_i}\|_{1,6B_i} \leq C_{Weak} \kappa 6r_i \|\nabla f\|_{1,\kappa 6B_i}$$

for all $B_i \in \mathcal{F}(B)$.

Proof. We can write:

$$\begin{aligned}
& \mu(6B_i \cap 6B_{i+1}) |f_{6B_i} - f_{6B_{i+1}}| \\
&= \int_{6B_i \cap 6B_{i+1}} |f_{6B_i} - f_{6B_{i+1}}| d\mu(x) \\
&\leq \int_{6B_i \cap 6B_{i+1}} |f(x) - f_{6B_i}| + |f(x) - f_{6B_{i+1}}| d\mu(x) \\
&\leq \int_{6B_i} |f(x) - f_{6B_i}| d\mu(x) + \int_{6B_{i+1}} |f(x) - f_{6B_{i+1}}| d\mu(x) \\
&\leq C_{Weak} 6\kappa \left(r_i \int_{6\kappa B_i} |\nabla f(x)| d\mu(x) + r_{i+1} \int_{6\kappa B_{i+1}} |\nabla f(x)| d\mu(x) \right) \\
&\leq C_{Weak} 6\kappa \left(r_i \int_{6\kappa B_i} |\nabla f(x)| d\mu(x) + 2r_i \int_{36\kappa B_i} |\nabla f(x)| d\mu(x) \right) \\
&\leq C_{Weak} 18\kappa r_i \int_{36\kappa B_i} |\nabla f(x)| d\mu(x).
\end{aligned}$$

This string of inequalities holds by the triangle inequality, set inclusion, the weak Poincaré inequality (Theorem 2.1.12), and the comparisons in Lemma 2.2.3.

By Lemma 2.2.3 we know that $\mu(B_i) \leq \mu(6B_i \cap 6B_{i+1})$, and so we can rewrite this to get:

$$\mu(B_i) |f_{6B_i} - f_{6B_{i+1}}| \leq 18\kappa r_i C_{Weak} \int_{36\kappa B_i} |\nabla f(x)| d\mu(x).$$

□

Recall that we have shown $C_{Weak} = \frac{2^{3n+3} M^3 \kappa^{N+1}}{\alpha n}$, as in Theorem 2.1.12 for our small balls containing only one vertex.

We are now in a position to prove our main theorem.

Theorem 2.2.6. *Let E be set whose subsets satisfy volume doubling with constant C_{vol} . Suppose \mathcal{F} is a Whitney type cover of E and that f satisfies*

$$\|f - f_{6B_i}\|_{1,6B_i} \leq C_{Weak} \kappa 6r_i \|\nabla f\|_{1,\kappa 6B_i}$$

for all $B_i \in \mathcal{F}$. Then

$$\int_E |f(x) - f_E| d\mu(x) \leq P_0 r \int_E |\nabla f(x)| d\mu(x)$$

holds where $P_0 = \left(1 + 3C_{vol}^{1+\log_2(10^3\kappa+9)}\right) KC_{Weak} 12\kappa 10^{-3}$.

Proof. We want to bound $|f - f_E|$. In order to do this, we will split this quantity into two essentially similar pieces, $|f - f_{6B_z}|$ and $|f_E - f_{6B_z}|$. At the end of the proof, we will show that $|f_E - f_{6B_z}|$ can be bounded by $|f - f_{6B_z}|$. Because of this, we only need consider $|f - f_{6B_z}|$. We will take f minus its average on the central ball and put it into a form where we can take advantage of the covering. This will involve splitting this further into chains of sufficiently small balls, and then applying the weak Poincaré inequality to them. After a bit of work, this will give us the desired inequality.

First, we will use the fact that $\cup_{B \in \mathcal{F}} 2B$ covers all of E to split the integral up into pieces.

$$\begin{aligned} \int_E |f(x) - f_{6B_z}| d\mu(x) &\leq \sum_{B \in \mathcal{F}} \int_{2B} |f(x) - f_{6B_z}| d\mu(x) \\ &\leq \sum_{B \in \mathcal{F}} \int_{2B} |f(x) - f_{6B}| d\mu(x) + \int_{2B} |f_{6B} - f_{6B_z}| d\mu(x). \end{aligned}$$

The first piece can be bounded nicely using the weak Poincaré inequality (Theorem 2.1.12).

$$\begin{aligned} \sum_{B \in \mathcal{F}} \int_{2B} |f(x) - f_{6B}| d\mu(x) &\leq \sum_{B \in \mathcal{F}} \int_{6B} |f(x) - f_{6B}| d\mu(x) \\ &\leq \sum_{B \in \mathcal{F}} C_{Weak} 6\kappa r_B \int_{6\kappa B} |\nabla f(x)| d\mu(x) \\ &\leq KC_{Weak} 6\kappa 10^{-3} r_E \int_E |\nabla f(x)| d\mu(x). \end{aligned}$$

The last part of the inequality follows from the fact that $6\kappa B \subset E$ (by Lemma 2.2.2), and at most K balls in $6\kappa\mathcal{F}$ overlap any given point in E .

The second piece can be rewritten as:

$$\begin{aligned} \sum_{B \in \mathcal{F}} \int_{2B} |f_{6B} - f_{6B_z}| d\mu(x) &= \sum_{B \in \mathcal{F}} \mu(2B) |f_{6B} - f_{6B_z}| \\ &\leq \sum_{B \in \mathcal{F}} C_{vol} \mu(B) |f_{6B} - f_{6B_z}|. \end{aligned}$$

Now let us consider what happens when we fix B . We have a chain, $\mathcal{F}(B)$, connecting B to the central ball; we can use this and Lemma 2.2.5 to find:

$$\begin{aligned} |f_{6B} - f_{6B_z}| &\leq \sum_{i=0}^{\ell-1} |f_{6B_i} - f_{6B_{i+1}}| \\ &\leq \sum_{i=0}^{\ell-1} C_{Weak} 18\kappa \frac{r_i}{\mu(B_i)} \int_{36\kappa B_i} |\nabla f(x)| d\mu(x) \\ &= \sum_{A \in \mathcal{F}(B)} C_{Weak} 18\kappa \frac{r_A}{\mu(A)} \int_{36\kappa A} |\nabla f(x)| d\mu(x). \end{aligned}$$

By lemma 2.2.4 we know that $B \subset (10^3\kappa + 9)A$ for any $A \in \mathcal{F}(B)$, and so we have $\chi_B = \chi_B \chi_{(10^3\kappa+9)A}$. Multiplying the previous inequality by this, summing over the B , and then integrating over E gives us:

$$\begin{aligned} &\int_E \sum_{B \in \mathcal{F}} |f_{6B} - f_{6B_z}| \chi_B(y) d\mu(y) \\ &\leq \int_E \sum_{B \in \mathcal{F}} \sum_{A \in \mathcal{F}(B)} C_{Weak} 18\kappa \frac{r_A}{\mu(A)} \int_{36\kappa A} |\nabla f(x)| d\mu(x) \chi_B(y) \chi_{(10^3\kappa+9)A}(y) d\mu(y). \end{aligned}$$

Since the B are disjoint, we have $\sum_{B \in \mathcal{F}} \chi_B(y) \leq 1$. This allows us to simplify the right hand side. We can then integrate.

$$\begin{aligned} \dots &\leq \int_E \sum_{A \in \mathcal{F}} C_{Weak} 18\kappa \frac{r_A}{\mu(A)} \int_{36\kappa A} |\nabla f(x)| d\mu(x) \chi_{(10^3\kappa+9)A}(y) d\mu(y) \\ &= \sum_{A \in \mathcal{F}} C_{Weak} 18\kappa \frac{r_A \mu((10^3\kappa + 9)A)}{\mu(A)} \int_{36\kappa A} |\nabla f(x)| d\mu(x). \end{aligned}$$

Volume doubling gives us:

$$\leq \sum_{A \in \mathcal{F}} C_{Weak} 18\kappa r_A C_{vol}^{\log_2(10^3\kappa+9)} \int_{36\kappa A} |\nabla f(x)| d\mu(x).$$

We then use the bound from (4) to see:

$$\leq C_{Weak} 18\kappa 10^{-3} r_E C_{vol}^{\log_2(10^3\kappa+9)} K \int_E |\nabla f(x)| d\mu(x).$$

Putting all of this together and factoring, our original inequality becomes:

$$\begin{aligned} \int_E |f(x) - f_{6B_z}| d\mu(x) \\ \leq \left(1 + 3C_{vol}^{1+\log_2(10^3\kappa+9)}\right) K C_{Weak} 6\kappa 10^{-3} r_E \int_E |\nabla f(x)| d\mu(x). \end{aligned}$$

Let $\frac{1}{2}P_0 = \left(1 + 3C_{vol}^{1+\log_2(10^3\kappa+9)}\right) K C_{Weak} 6\kappa 10^{-3}$. Then we can rewrite the inequality as:

$$\int_E |f(x) - f_{6B_z}| d\mu(x) \leq \frac{1}{2} P_0 r_E \int_E |\nabla f(x)| d\mu(x).$$

All that remains is to switch from f_{6B_z} to the average on the entire set, f_E .

$$\begin{aligned} \int_E |f_E - f_{6B_z}| d\mu(x) &= \mu(E) |f_E - f_{6B_z}| \\ &= \mu(E) \left| \frac{1}{\mu(E)} \int_E f(x) - f_{6B_z} d\mu(x) \right| \\ &\leq \int_E |f(x) - f_{6B_z}| d\mu(x) \\ &\leq \frac{1}{2} P_0 r_E \int_E |\nabla f(x)| d\mu(x). \end{aligned}$$

Thus, the Poincaré inequality holds on the ball $E = B(z, r)$.

$$\begin{aligned} \int_E |f(x) - f_E| d\mu(x) &\leq \int_E |f(x) - f_{6B_z}| d\mu(x) + \int_E |f_{6B_z} - f_E| d\mu(x) \\ &\leq P_0 r_E \int_E |\nabla f(x)| d\mu(x). \end{aligned}$$

□

Corollary 2.2.7. *Let X be an admissible n -dimensional Euclidean complex with degree bounded above by M , solid angle bounded by α , and edge lengths bounded below by ℓ . Let $E = B(z, r)$ where $r < R_0 := \frac{\ell}{\kappa}$. Then*

$$\int_E |f(x) - f_E| d\mu(x) \leq P_0 r \int_E |\nabla f(x)| d\mu(x)$$

holds for $f \in \text{Lip}(X) \cap L^1(E)$ where $\kappa = 6(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1)^n$ and $P_0 = (1 + 3M^32^n(10^3\kappa + 9)^n)M(8(1 + 10^3\kappa))^nC_{Weak}6\kappa$.

Proof. Apply Theorem 2.2.6 with $K = M(8(1 + 10^3\kappa))^n$ and $C_{vol} = M2^n$. \square

Corollary 2.2.8. *Let X be an admissible n -dimensional Euclidean complex with degree bounded above by M , solid angle bounded by α , and edge lengths bounded below by ℓ . For $f \in \text{Lip}(X) \cap L^p(E)$ and $r < R_0$ we have*

$$\inf_c ||f - c||_{p,E} \leq pP_0r||\nabla f||_{p,E}$$

where E is a ball of radius r and $1 \leq p < \infty$. Note that this implies:

$$||f - f_E||_{p,E} \leq 2pP_0r||\nabla f||_{p,E}.$$

Here $P_0 = (1 + 3M^32^n(10^3\kappa + 9)^n)M(8(1 + 10^3\kappa))^nC_{Weak}6\kappa$, $C_{Weak} = \frac{2^{3n+3}M^3\kappa^{n+1}}{\alpha n}$, and $\kappa = 6(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1)^n$.

Proof. Apply Lemma 2.1.9 to Corollary 2.2.7. \square

Corollary 2.2.9. *Assume $p = 1$ Poincaré inequality $||f - f_B||_{1,B} \leq P_0r||\nabla f||_{1,B}$ holds for $f \in \text{Lip}(X) \cap L^p(E)$ on balls $B = B(x, r)$ with $r \leq R$. Assume volume doubling holds with constant C_{vol} for balls with radius less than C_0R . Then*

$$||f - f_E||_{p,E} \leq p2P_0 \left(6(1 + 3C_{vol}^{11})C_{vol}^{13}10^{-3}\right)^{\lceil \log_{\frac{10^3}{6}}(C_0) \rceil} r_E ||\nabla f||_{p,E}$$

also holds for balls E with radius less than C_0R and $1 \leq p < \infty$.

Proof. Note that if $E = B(x, \frac{1}{6}10^3r)$, then the $p = 1$ Poincaré inequality holds for all balls in the Whitney cover dilated by a factor of 6 with $\kappa = 1$. We can apply Theorem 2.2.6 which gives us

$$||f - f_E||_{1,E} \leq 6(1 + 3C_{vol}^{11})C_{vol}^{13}10^{-3}P_0r_E||\nabla f||_{1,E}.$$

In particular, we can repeat this to show that the $p = 1$ Poincaré inequality holds for balls up to radius $C_0 R$ with constant $(6(1 + 3C_{vol}^{11})C_{vol}^{13}10^{-3})^{\lceil \log_{10^3}(C_0) \rceil} P_0$. To get the p Poincaré inequality, apply lemma 2.1.9. \square

Note that bounds on degree M , angles α and edge lengths ℓ give us uniform local volume doubling on our complex, X . For any fixed R we can then apply Lemma 2.1.9 to Corollary 2.2.9 to get the standard L^2 Poincaré inequality for balls of radius up to R . In general, this cannot be extended to $R = \infty$; note that the constant in the new Poincaré inequality goes to infinity as C_0 goes to infinity.

Chapter 3

Small Time Heat Kernel Estimates for X

The heat kernel, $h_t(x, y)$, is the fundamental solution to the heat equation

$$\partial_t u = \Delta u.$$

Note that our formulation does not have factors of -1 or $\frac{1}{2}$, which appear in some of the literature. This type of differential equation is parabolic; one way of obtaining information about it is through parabolic Harnack inequalities. Sturm [29] shows that local volume doubling and Poincaré inequalities on a subset of a complete metric space imply a local parabolic Harnack inequality on that subset. He then uses this to find Gaussian estimates on the heat kernel. The equivalence of the parabolic Harnack inequality with Poincaré and volume doubling had previously been done in the Riemannian manifold case by Grigor'yan [15] and Saloff-Coste [27].

3.1 Small time Heat Kernel Asymptotics

We have shown a uniform local Poincaré inequality, and our complex is both complete and locally satisfies volume doubling. This tells us that we've satisfied the hypotheses of the following theorem of Sturm [29] which gives a lower bound on the diagonal.

Theorem 3.1.1 (Sturm). *Assume Y is an open subset of a complete space X that admits a Poincaré inequality with constant C_P and volume doubling with constant 2^N . Then there exists a constant $C = C(C_P, N)$ such that*

$$h_t(x, x) \geq \frac{1}{C\mu(B(x, \sqrt{t}))}$$

for all $x \in Y$ and all t such that $0 < t < \rho^2(x, X - Y)$.

Here, ρ refers to the intrinsic distance. In our complex, this will always satisfy $\rho(x, y) \geq d(x, y)$. Also note that since we have a uniform local Poincaré inequality and a uniform local volume doubling constant we have the following corollary:

Corollary 3.1.2. *Let X be an admissible n -dimensional Euclidean complex with degree bounded above by M , solid angle bounded by α , and edge lengths bounded below. For any $R_0 > 0$ there is a corresponding constant $C = C(X, R_0)$ so that*

$$h_t(x, x) \geq \frac{1}{CM\mu(S^{(n-1)})t^{n/2}}$$

for all $x \in X$ and all t such that $0 < t < R_0^2$.

Proof. For each $x \in X$ apply 3.1.1 to X with $Y = B(x, R_0)$. The distance compares easily: $\rho(x, X - Y) \geq d(x, X - Y) = R_0$. Because the constant C in 3.1.1 depends only on C_P and N , we can use the fact that our constants C_P and N do depend only on the radius of our ball to obtain a universal constant, C . \square

Sturm [29] also proves an upper bound for the heat kernel; this bound is especially useful near the diagonal.

Theorem 3.1.3 (Sturm). *Assume Y is an open subset of a complete space X that admits a Poincaré inequality with constant C_P and volume doubling with constant 2^N . Then there exists a constant $C = C(C_P, N)$ such that for every $x, y \in Y$*

$$h_t(x, y) \leq \frac{Ce^{-\frac{\rho^2(x, y)}{4t}}}{\sqrt{\mu(B(x, \sqrt{T}))\mu(B(y, \sqrt{T}))}} \left(1 + \frac{\rho^2(x, y)}{t}\right)^{N/2} e^{-\lambda t} (1 + \lambda t)^{1+N/2}$$

where ρ is the intrinsic distance, $R = \inf(\rho(x, X - Y), \rho(y, X - Y))$, $T = \min(t, R^2)$ and λ is the bottom of the spectrum of the self-adjoint operator $-L$ on $L^2(X, \mu)$.

In general, we can replace λ with 0, which increases the value of the right hand side. In our setting, we can simplify this a bit more.

Corollary 3.1.4. *Let X be an admissible n -dimensional Euclidean complex with degree bounded above by M , solid angle bounded by α , and edge lengths bounded below. Then for any R_0 there exists a constant $C = C(X, R_0)$ so that for any $x \in X$ and $t > 0$ we have:*

$$h_t(x, x) \leq \frac{C}{M\mu(S^{(n-1)})(\min(t, R_0^2))^{n/2}}.$$

Proof. For each $x \in X$ apply 3.1.3 to X with $Y = B(x, R_0)$. The distance compares easily: $\rho(x, X/Y) \geq d(x, X/Y) = R_0$. The $d(x, x)$ terms drop out, as do the λ terms. Because the constant C in 3.1.3 depends only on C_P and N , we can use the fact that our constants C_P and N do not depend on our specific choice of ball to obtain a universal constant, C . \square

Corollary 3.1.5. *Let X be an admissible n -dimensional Euclidean complex with degree bounded above by M , solid angle bounded by α , and edge lengths bounded below. For any R_0 there exists a $C = C(X, R_0)$ so that for any $x, y \in X$ and $t > 0$ we have:*

$$h_t(x, y) \leq \frac{C}{M\mu(S^{(n-1)})(\min(t, R_0^2))^{n/2}} e^{-\frac{d^2(x, y)}{4t}} \left(1 + \frac{d^2(x, y)}{t}\right)^{N/2}.$$

Proof. For each $x, y \in X$ apply 3.1.3 to X with $Y = B(x, R_0) \cup B(y, R_0)$. The distance compares easily: $\rho(x, X/Y) \geq d(x, X/Y) = R_0$. Because the constant C in 3.1.3 depends only on C_P and N , we can use the fact that our constants C_P and N do not depend on our specific choice of ball to obtain a universal constant, C . \square

Note that we can rewrite this as a bound of the following form for some constants C, c :

$$h_t(x, y) \leq \frac{C}{(\min(t, R_0^2))^{n/2}} e^{-c \frac{d^2(x, y)}{t}}.$$

Corollary 3.1.6. *For an admissible n -dimensional Euclidean complex X with degree bounded above by M , solid angle bounded by α , and edge lengths bounded below, on $X^{(k)}$ we have*

$$\frac{1}{C_k t^{k/2}} \leq h_t^k(x, x) \leq \frac{C_k}{t^{k/2}}.$$

This holds for all $t < R_0^2$ and $x \in X^{(k)}$, where C_k depends on R_0, α, M, k , and $\inf_{v, w \in X^{(0)}} d(v, w)$. In particular, we can take $C = \max_{k=1..n} C_k$ to have a uniform constant for each $X^{(k)}$.

Proof. Apply Corollaries 3.1.2 and 3.1.4 to $X^{(k)}$. This holds because $X^{(k)}$ is also an admissible complex satisfying the same bounds as X . C_k varies slightly in each dimension due to the effect of dimension on volume doubling, and hence Poincaré. \square

Off diagonal, the lower bound is more complicated.

Theorem 3.1.7 (Sturm). *Assume Y is an open subset of a complete space X that admits a Poincaré inequality with constant C_P and volume doubling with constant 2^N . Then there exists a constant $C = C(C_P, N)$ such that for every $x, y \in Y$ which are joined by a curve γ of length $\rho(x, y)$*

$$h_t(x, y) \geq \frac{1}{C \mu(B(x, \sqrt{T}))} e^{-C \frac{\rho^2(x, y)}{t}} e^{-\frac{Ct}{R^2}}$$

where ρ is the intrinsic distance, $R = \inf_{0 \leq s \leq 1} (\rho(\gamma(s), X - Y))$, and $T = \min(t, R^2)$.

In our setting, we can find a near-diagonal lower bound for any complex.

Corollary 3.1.8. *Let X be an admissible n -dimensional Euclidean complex with degree bounded above by M , solid angle bounded by α , and edge lengths bounded below. For any $R_0 > 0$ there exists a $C = C(X, R_0)$ so that for any $x, y \in X$ with $d(x, y) < R_0$ and $t > 0$ we have:*

$$h_t(x, y) \geq \frac{1}{C\mu(B(x, \sqrt{\min(t, R_0^2)}))} e^{-C\frac{d^2(x, y)}{t}} e^{-\frac{Ct}{R_0^2}}.$$

If X is volume doubling and has a global Poincaré inequality, we can set $R_0 = \infty$ to get:

$$h_t(x, y) \geq \frac{1}{C\mu(B(x, \sqrt{t}))} e^{-C\frac{d^2(x, y)}{t}}.$$

Proof. For each $x, y \in X$ apply 3.1.7 to X with $Y = B(x, 2R_0)$. Since we have a length space, the distance compares easily: $\rho(\gamma(s), X/Y) \geq d(\gamma(s), X/Y) = R_0$. Because the constant C in 3.1.3 depends only on C_P and N , we can use the fact that our constants C_P and N do not depend on our specific choice of ball to obtain a universal constant, C . \square

3.2 Examples

Example 3.2.1. In R^1 the heat kernel, $h_t(x, y)$ is the density for the transition probability of Brownian motion.

$$h_t(x, y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}.$$

This is a normal density for y with expectation x and variance $2t$. In the probability literature, it is common for the heat equation to be written with the time derivative multiplied by an extra factor of $1/2$ so that the variance is t . See for example Feller Volume 2 [13].

We can think of R^1 as an Euclidean complex. This matches our on diagonal bound exactly, but it is slightly nicer (by a factor of $\sqrt{1 + d^2(x, y)/t}$ than our off diagonal bound.

Example 3.2.2. In R^n , the heat equation can be solved using either a scaling argument or Fourier series. Alternately, it can be thought of as an n-dimensional version of Brownian motion. In the PDE literature, the heat kernel is also called the Gauss kernel or the fundamental solution to the heat equation. See Evans [12] for a derivation.

$$h_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}.$$

Note that R^n is also an Euclidean complex, and that this kernel is consistent with our asymptotics.

Example 3.2.3. We can think of a circle of length 1 as a complex consisting of three edges of length 1/3 joined in a triangle shape. One can calculate the heat kernel in terms of a sum using Fourier series; see Dym and McKean [10]. The heat kernel here is

$$h_t(x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{|x-y-n|^2}{4t}}.$$

When $x = y$, this simplifies to:

$$h_t(x, x) = \frac{1}{\sqrt{4\pi t}} + \frac{2}{\sqrt{4\pi t}} \sum_{n=1}^{\infty} e^{-\frac{n^2}{4t}}.$$

For small values of t , the dominant term is $\frac{1}{\sqrt{4\pi t}}$. This is the same behavior as our small time prediction. Note that once $t = 1/4$, the ball of radius \sqrt{t} will be of size 1. By this point in time, the asymptotic will cease to be useful.

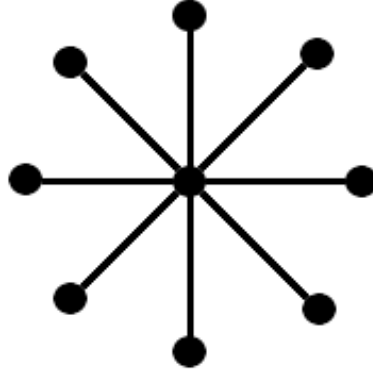


Figure 3.1: Example of a star; here $n=8$.

Example 3.2.4. We will look at the heat kernel on a star shaped graph, X , which has a central vertex with n edges attached to it.

When we compute functions on X for $x \in e(a, b)$, we will let x represent $d(x, a)$. For example, a would be 0, the midpoint would be $\frac{1}{2}$, and b would be 1. All functions on X are of the form $f(x, j) = \sum_i f_i(x) I_{j=i} + f(0) I_{x=0}$ where $x \in (0, 1]$ and $j = 1, 2, \dots, n$. $f_i(x)$ represents the value of the function along the i th leg, and $f(0)$ is the value at the center of the star. The measure on our star is $d\mu(x, j) = dx$. When we write the derivative $\frac{df}{dx}$ we will mean the usual derivative with respect to Lebesgue measure. $\frac{df}{d\mu}(0) = \sum_i \frac{df_i}{dx}(0)$ for functions f which are differentiable for each f_i on $(0, 1]$.

We'll look at a domain where our function has zero derivative at the boundary points and has zero derivative in the center.

$\text{Dom}(\Delta) = \{f \in C(X) : f_i \in C^1((0, 1]) \text{ , and } \frac{df}{d\mu}(0) = 0, \frac{df_i}{d\mu}(1) = 0\}$; we will be using an L^2 norm on this space.

The symmetric set of eigenfunctions are cosine on every leg:

$$\Phi_k(x) = \sqrt{\frac{2}{n}} \cos(k\pi x) \text{ for } x \in e_i, \ i = 1..n,$$

or, when $k = 0$, they are a constant on every leg:

$$\Phi_0(x) = \frac{1}{\sqrt{n}} \text{ for } x \in e_i, \ i = 1..n.$$

These functions have derivative zero at each vertex and are continuous at the central vertex. The coefficients are chosen so that they have an L^2 norm of 1.

Note that if we look at the product of these for points x and y on the star (regardless of which leg they occur on), we have

$$\begin{aligned} \Phi_k(x)\Phi_k(y) &= \frac{2}{n} \cos(k\pi x) \cos(k\pi y) \\ &= \frac{1}{n} (\cos(k\pi(x-y)) + \cos(k\pi(x+y))). \end{aligned}$$

If we sum $e^{-\lambda^2 t} \Phi(x)\Phi(y)$ we find:

$$\begin{aligned} \frac{1}{n} + \frac{1}{n} \sum_{k=1}^{\infty} e^{-(k\pi)^2 t} (\cos(k\pi(x-y)) + \cos(k\pi(x+y))) \\ = \frac{1}{2n} \sum_{k=-\infty}^{\infty} e^{-(k\pi)^2 t} (\cos(k\pi(x-y)) + \cos(k\pi(x+y))). \end{aligned}$$

In order to simplify this, we will use Jacobi's identity (see Dym for derivation):

$$\sum_{k=-\infty}^{\infty} e^{-\frac{(x-k)^2}{2s}} = \sqrt{2\pi s} \sum_{k=-\infty}^{\infty} e^{-2\pi^2 k^2 s} e^{2\pi i k x}.$$

Because this sums to a real number, we can rewrite it as:

$$\sum_{k=-\infty}^{\infty} e^{-\frac{(x-k)^2}{2s}} = \sqrt{2\pi s} \sum_{k=-\infty}^{\infty} e^{-2\pi^2 k^2 s} \cos(2\pi k x).$$

This gives us:

$$\begin{aligned} \frac{1}{2n} \sum_{k=-\infty}^{\infty} e^{-(k\pi)^2 t} (\cos(k\pi(x-y)) + \cos(k\pi(x+y))) \\ = \frac{1}{2n\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} \left(e^{-\frac{(x-y-2k)^2}{4t}} + e^{-\frac{(x+y-2k)^2}{4t}} \right). \end{aligned}$$

We will also have ones that form an $n-1$ dimensional basis on the legs. These are the “odd” eigenfunctions. These will be either sine or 0 along the legs. Since

$\sin(0) = 0$, they will be continuous at the center. We require the derivatives at the vertices to be 0, and so the possible sine functions are $\sin\left(\frac{(2k+1)\pi}{2}x\right)$. Note that they must be normalized according to an L^2 norm; this means that $\sum_{i=1}^n \frac{1}{2}b_i^2 = 1$, where the b_i are the coefficients. Since they will need to have derivative zero at the central vertex, we will need $\sum_{i=1}^n b_i = 0$. The combination of these two restrictions, along with the fact that these eigenfunctions must be orthogonal to one another, determine the coefficients.

For $i = 1 \dots \lfloor \frac{n}{2} \rfloor$ we have eigenfunctions of the form:

$$\tilde{\Phi}_{k,i}(x) = \begin{cases} \sin\left(\frac{(2k+1)\pi}{2}x\right) & \text{for } x \in e_{2i-1} \\ -\sin\left(\frac{(2k+1)\pi}{2}x\right) & \text{for } x \in e_{2i} \\ 0 & \text{otherwise.} \end{cases}$$

These functions are trivially orthogonal to one another. The factors of ± 1 give us derivative 0 at the center.

For $i = 1 \dots \lfloor \frac{n}{2} \rfloor - 1$ we have:

$$\tilde{\Phi}_{k, \lfloor \frac{n}{2} \rfloor + i}(x) = \begin{cases} \frac{1}{\sqrt{i(i+1)}} \sin\left(\frac{(2k+1)\pi}{2}x\right) & \text{for } x \in e_j, j = 1..2i \\ -\frac{i}{\sqrt{i(i+1)}} \sin\left(\frac{(2k+1)\pi}{2}x\right) & \text{for } x \in e_j, j = 2i+1, 2i+2 \\ 0 & \text{otherwise.} \end{cases}$$

In the case where there is an odd number of legs we have:

$$\tilde{\Phi}_{k,n-1}(x) = \begin{cases} \frac{\sqrt{2}}{\sqrt{n(n-1)}} \sin\left(\frac{(2k+1)\pi}{2}x\right) & \text{for } x \in e_j, j = 1..n-1 \\ -(n-1) \frac{\sqrt{2}}{\sqrt{n(n-1)}} \sin\left(\frac{(2k+1)\pi}{2}x\right) & \text{for } x \in e_n \end{cases}$$

Note that edges $2i$ and $2i-1$ have constants with the same sign, which forces the functions to be orthogonal to the first set. The pattern of $+$ and $-$ allow them to be orthogonal to one another. The other factors guarantee that the derivative at the center is zero.

If we look at the product of the $\tilde{\Phi}_k$ for points x and y on the star, we have

$$\begin{aligned}\tilde{\Phi}_k(x)\tilde{\Phi}_k(y) &= c \sin\left(\frac{(2k+1)\pi}{2}x\right) \sin\left(\frac{(2k+1)\pi}{2}y\right) \\ &= \frac{c}{2} \left(\cos\left((2k+1)\pi\frac{x-y}{2}\right) - \cos\left((2k+1)\pi\frac{x+y}{2}\right) \right).\end{aligned}$$

If we sum these “odd” eigenfunctions, multiplied by $e^{-\frac{(2k+1)^2\pi^2}{4}t}$, we have:

$$\sum_{k=0}^{\infty} e^{-\frac{(2k+1)^2\pi^2}{4}t} \frac{c}{2} \left(\cos\left((2k+1)\pi\frac{x-y}{2}\right) - \cos\left((2k+1)\pi\frac{x+y}{2}\right) \right).$$

We can rewrite the $x-y$ terms as follows; a similar calculation will work for the $x+y$ terms. First we add in terms with $2k$. We note that cosine is an even function, and so we can extend this to negative k . We also have the zero term, since it will cancel between the two sums.

$$\begin{aligned}& \sum_{k=0}^{\infty} e^{-((2k+1)\pi)^2\frac{t}{4}} \frac{c}{2} \cos\left((2k+1)\pi\frac{x-y}{2}\right) \\ &+ \sum_{k=1}^{\infty} e^{-(2k\pi)^2\frac{t}{4}} \frac{c}{2} \cos\left(2k\pi\frac{x-y}{2}\right) - \sum_{k=1}^{\infty} e^{-(2k\pi)^2\frac{t}{4}} \frac{c}{2} \cos\left(2k\pi\frac{x-y}{2}\right) \\ &= \sum_{k=1}^{\infty} e^{-(k\pi)^2\frac{t}{4}} \frac{c}{2} \cos\left(k\pi\frac{x-y}{2}\right) - \sum_{k=1}^{\infty} e^{-(k\pi)^2t} \frac{c}{2} \cos(k\pi(x-y)) \\ &= \sum_{k=-\infty}^{\infty} e^{-(k\pi)^2\frac{t}{4}} \frac{c}{4} \cos\left(k\pi\frac{x-y}{2}\right) - \sum_{k=-\infty}^{\infty} e^{-(k\pi)^2t} \frac{c}{4} \cos(k\pi(x-y)).\end{aligned}$$

We will apply Jacobi’s identity to each of the sums. The first sum yields:

$$\begin{aligned}\frac{c}{4} \sum_{k=-\infty}^{\infty} e^{-(k\pi)^2\frac{t}{4}} \cos\left(2k\pi\frac{x-y}{4}\right) &= \frac{c}{4\sqrt{2\pi\frac{t}{8}}} \sum_{k=-\infty}^{\infty} e^{-\frac{(\frac{x-y}{4}-k)^2}{\frac{2t}{8}}} \\ &= \frac{c}{2\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(x-y-4k)^2}{4t}}.\end{aligned}$$

The second sum gives us:

$$-\frac{c}{4} \sum_{k=-\infty}^{\infty} e^{-(k\pi)^2t} \cos\left(2k\pi\frac{x-y}{2}\right) = \frac{c}{4\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} -e^{-\frac{(x-y-2k)^2}{4t}}.$$

Similarly, for the $x + y$ terms we have:

$$\begin{aligned} -\sum_{k=1}^{\infty} e^{-\frac{(2k+1)^2\pi^2}{4}t} \frac{c}{2} \cos\left((2k+1)\pi\frac{x+y}{2}\right) \\ = -\frac{c}{2\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(x+y-4k)^2}{4t}} + \frac{c}{4\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(x+y-2k)^2}{4t}}. \end{aligned}$$

To get the heat kernel, we sum the $e^{-\lambda^2 t} \Phi(x) \Phi(y)$:

$$\begin{aligned} h_t(x, y) &= \frac{1}{2\sqrt{\pi t} n} \sum_{k=-\infty}^{\infty} \left(e^{-\frac{(x-y-2k)^2}{4t}} + e^{-\frac{(x+y-2k)^2}{4t}} \right) \\ &\quad + \frac{c}{2\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} \left(e^{-\frac{(x-y-4k)^2}{4t}} - e^{-\frac{(x+y-4k)^2}{4t}} \right) \\ &\quad + \frac{c}{4\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} \left(e^{-\frac{(x+y-2k)^2}{4t}} - e^{-\frac{(x-y-2k)^2}{4t}} \right). \end{aligned}$$

Note that the value of c depends on which edges x and y are on. If they are on the same edge, $c = 2\left(1 - \frac{1}{n}\right)$. If x and y are on different edges, then $c = -\frac{2}{n}$.

We are now in a position to see what happens on the star near $t = 0$. On the diagonal, the heat kernel will limit to infinity as t goes to zero with the following asymptotics:

$$\begin{aligned} h_t(0, 0) &\approx \frac{1}{n\sqrt{\pi t}}, \\ h_t(1, 1) &\approx \frac{1}{\sqrt{\pi t}}, \text{ and} \\ h_t(x, x) &\approx \frac{1}{2\sqrt{\pi t}} \text{ for } x \neq 0, 1. \end{aligned}$$

To determine what happens for t near zero when we have two different points, we need to consider the relative positions of x and y . We know that the heat kernel will limit to zero as t goes to zero. Without loss of generality, we will look at when $d(x, 0) < d(y, 0)$.

When $x = 0$ and $y \neq 1$, the $k = 0$ terms will dominate. They give us:

$$h_t(0, y) \approx \frac{1}{n\sqrt{\pi t}} e^{-\frac{y^2}{4t}}.$$

When $x = 0$ and $y = 1$, we will only have terms involving $e^{-\frac{1}{4t}}$ contributing.

After cancellation, this gives us:

$$h_t(0, 1) \approx \frac{2}{n\sqrt{\pi t}} e^{-\frac{1}{4t}}.$$

When $x \neq 0$, the dominant terms will involve $e^{-\frac{(x-y)^2}{4t}}$. If the coefficient for those terms is zero, we then have $e^{-\frac{(x+y)^2}{4t}}$ instead. The relevant terms are:

$$\begin{aligned} h_t(x, 1) &\approx \left(\frac{2}{n} - c\right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x+1)^2}{4t}} + \left(\frac{2}{n} + c\right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-1)^2}{4t}}, \text{ and} \\ h_t(x, y) &\approx \left(\frac{2}{n} - c\right) \frac{1}{4\sqrt{\pi t}} e^{-\frac{(x+y)^2}{4t}} + \left(\frac{2}{n} + c\right) \frac{1}{4\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} \text{ for } x \neq y. \end{aligned}$$

When x and y are on different edges, $c = -\frac{2}{n}$, and so all of the terms with coefficient $\frac{2}{n} + c$ disappear. Note that the notation gives us $d(x, y) = x + y$:

$$\begin{aligned} h_t(x, 1) &\approx \frac{2}{n\sqrt{\pi t}} e^{-\frac{(x+1)^2}{4t}} \text{ for } x \neq 1, \text{ and} \\ h_t(x, y) &\approx \frac{1}{n\sqrt{\pi t}} e^{-\frac{(x+y)^2}{4t}} \text{ for } x \neq y, y \neq 1. \end{aligned}$$

When x and y are on the same edge, $\frac{2}{n} + c = \frac{2}{n} + 2\left(1 - \frac{1}{n}\right) = 2$. This gives us the same asymptotic as in **R**.

$$\begin{aligned} h_t(x, 1) &\approx \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-1)^2}{4t}}, \text{ and} \\ h_t(x, y) &\approx \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} \text{ for } x \neq y. \end{aligned}$$

Example 3.2.5. Consider the graph consisting of two central vertices which are joined by one edge with edges coming off of them. We will calculate the behaviour of the heat kernel between the two central vertices for small times.

Consider two star-like graphs joined together by a central edge, $e(v_1, v_2)$. The star centered at v_1 connects to n edges in addition to the central edge. These are labeled $e(v_1, w_i)$ $i = 1..n$. For the star centered at v_2 , there are m such edges

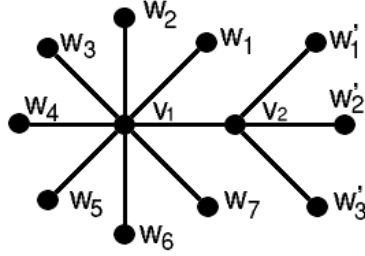


Figure 3.2: Example of two joined stars with labels; here $n=7$ and $m=3$.

which are labeled $e(v_2, w'_i)$ $i = 1..m$. This graph has three different types of eigenfunctions. When we compute these functions for $x \in e(a, b)$, we will let x represent $d(x, a)$. So a would be 0, the midpoint would be $\frac{1}{2}$, and b would be 1.

The first kind correspond to eigenvalues of the form $\frac{(2k+1)^2\pi^2}{4}$. They form an $m-1+n-1$ dimensional space. The corresponding eigenfunctions are of the form $c(e(v, w)) \sin(\frac{(2k+1)\pi}{2}x)$ on the edges $e(v_1, w_i)$ and $e(v_2, w'_i)$ and are 0 on the central edge, $e(v_1, v_2)$. Since each eigenfunction is zero on the central edge, they will not contribute when we compute $h_t(x, y)$ for $x \in e(v_1, v_2)$.

The second kind correspond to eigenvalues of the form $(k\pi)^2$. When $k \neq 0$, these are $\pm \frac{\sqrt{2}}{\sqrt{m+n+1}} \cos(k\pi x)$ along each edge in the graph, with sign chosen to preserve continuity. We write them:

$$\Phi_k(x) = \begin{cases} \frac{\sqrt{2}}{\sqrt{m+n+1}} \cos(k\pi x) & \text{for } x \in e(v_1, v_2) \text{ or } e(v_1, w_i) \ i = 1..n \\ \text{sgn}(\cos(k\pi)) \frac{\sqrt{2}}{\sqrt{m+n+1}} \cos(k\pi x) & \text{for } x \in e(v_2, w'_i) \ i = 1..m. \end{cases}$$

The product of this function with itself at the vertices v_1 and v_2 is

$$\Phi_k(v_1)\Phi_k(v_2) = \frac{2}{m+n+1} \cos(k\pi).$$

When $k = 0$, we have $\Phi_0(v_1)\Phi_0(v_2) = \frac{1}{m+n+1}$.

If we look at the sum of $\Phi_k(v_1)\Phi_k(v_2)e^{-k^2\pi^2t}$, we can write it as follows:

$$\begin{aligned}\sum_{k=0}^{\infty} \Phi_k(v_1)\Phi_k(v_2)e^{-k^2\pi^2t} &= \frac{1}{m+n+1} + \sum_{k=1}^{\infty} \frac{2}{m+n+1} \cos(k\pi)e^{-k^2\pi^2t} \\ &= \frac{1}{m+n+1} \sum_{k=-\infty}^{\infty} \cos(k\pi)e^{-k^2\pi^2t}.\end{aligned}$$

The third kind are more complicated. They are of the form

$$f(x) = c_1(e) \left(\sin \left(\sqrt{\lambda}(1-x) \right) + c_2(e) \sin \left(\sqrt{\lambda}x \right) \right)$$

along each edge, where $c_1(e)$ and $c_2(e)$ will depend on the edge e ; see [24]. For these to be in the domain, they must be continuous and have zero derivative at each vertex. We have $2(m+n+1)$ constants $c_i(e)$ to determine, as well as the possible values of λ .

To guarantee continuity, we need the function to have the same value at v_1 regardless of which edge we're considering. To get this, we set

$$c_1(e(v_1, w_i)) = c_1(e(v_1, v_2)).$$

Similarly, for continuity at v_2 , we need

$$c_1(e(v_2, w'_i)) = c_1(e(v_1, v_2))c_2(e(v_1, v_2)).$$

This brings us down to $m+n+2$ different $c_i(e)$ that we need to determine.

Now we need zero derivative at the vertices. Note that for an edge e

$$f'(x) = \sqrt{\lambda}c_1(e) \left(-\cos \left(\sqrt{\lambda}(1-x) \right) + c_2(e) \cos \left(\sqrt{\lambda}x \right) \right).$$

For edges of the form $e(v, w)$, we find that setting $c_2(e(v, w)) = \frac{1}{\cos(\sqrt{\lambda})}$ will allow f to satisfy $f'(w) = 0$. For $f'(v_1) = 0$, we need $c_2(v_1, v_2) = \frac{(n+1)\cos^2(\sqrt{\lambda})-n}{\cos(\sqrt{\lambda})}$. We now have only two things that can be used to determine our function; $c_1(e(v_1, v_2))$

and λ . Varying the value of $c_1(e(v_1, v_2))$ will multiply the entire function by a constant. We'll need to determine λ in order to have $f'(v_2) = 0$. For this, we need to solve

$$\begin{aligned} \sqrt{\lambda} c_1(e(v_1, v_2)) \left(m \frac{(n+1) \cos^2(\sqrt{\lambda}) - n}{\cos(\sqrt{\lambda})} \left(-\cos(\sqrt{\lambda}) + \frac{1}{\cos(\sqrt{\lambda})} \right) \right. \\ \left. + (-1) \left(-1 + \frac{(n+1) \cos^2(\sqrt{\lambda}) - n}{\cos(\sqrt{\lambda})} \cos(\sqrt{\lambda}) \right) \right) = 0. \end{aligned}$$

The (-1) comes from the fact that we want the sum of the inward pointing derivatives along the edges to sum to zero. We can simplify this to:

$$m \left(-(n+1) \cos^2(\sqrt{\lambda}) + 2n + 1 - \frac{n}{\cos^2(\sqrt{\lambda})} \right) + \left(n + 1 - (n+1) \cos^2(\sqrt{\lambda}) \right) = 0.$$

This can be written as a quadratic equation in $\cos^2(\sqrt{\lambda})$; when solved, the only possibility for $\cos^2(\sqrt{\lambda})$ in $[0, 1]$ is $\cos^2(\sqrt{\lambda}) = \frac{mn}{(m+1)(n+1)}$.

Finally, we set $c_1(e(v_1, v_2))$ so that $\|f\|_2 = 1$. This holds when

$$c_1(e(v_1, v_2)) = \frac{\sqrt{m(m+1)}}{m+n+1}.$$

Putting all of this information together, we find that the third type of eigenfunction has the form:

$$\tilde{\Phi}_k(x) = \begin{cases} \frac{\sqrt{m(m+1)}}{m+n+1} \left(\sin(\sqrt{\lambda_k}(1-x)) + \frac{(n+1) \cos^2(\sqrt{\lambda_k}) - n}{\cos(\sqrt{\lambda_k})} \sin(\sqrt{\lambda_k}x) \right) & \text{for } x \in e(v_1, v_2) \\ \frac{\sqrt{m(m+1)}}{m+n+1} \left(\sin(\sqrt{\lambda_k}(1-x)) + \frac{1}{\cos(\sqrt{\lambda_k})} \sin(\sqrt{\lambda_k}x) \right) & \text{for } x \in e(v_1, w_i), i = 1..n \\ \frac{\sqrt{m(m+1)}((n+1) \cos^2(\sqrt{\lambda_k}) - n)}{\cos(\sqrt{\lambda_k})(m+n+1)} \left(\sin(\sqrt{\lambda_k}(1-x)) + \frac{1}{\cos(\sqrt{\lambda_k})} \sin(\sqrt{\lambda_k}x) \right) & \text{for } x \in e(v_2, w'_i), i = 1..m. \end{cases}$$

These eigenfunctions correspond to eigenvalues λ_k , where

$\cos^2(\sqrt{\lambda_k}) = \frac{mn}{(m+1)(n+1)}$. Let $\sqrt{\lambda_0}$ be the square root of the eigenvalue in $(0, \frac{\pi}{2})$.

All other $\sqrt{\lambda}$ are of the form $\sqrt{\lambda_0} + k\pi$ and $-\sqrt{\lambda_0} + k\pi$, where k is a positive integer. When k is even, the cosine is positive; for k odd, cosine is negative.

When we look at the product of $\tilde{\Phi}_k(v_1)$ and $\tilde{\Phi}_k(v_2)$, this expression simplifies greatly.

$$\tilde{\Phi}_k(v_1)\tilde{\Phi}_k(v_2) = -\operatorname{sgn}\left(\cos\left(\sqrt{\lambda_k}\right)\right) \frac{\sqrt{mn}}{\sqrt{(m+1)(n+1)}(m+n+1)}.$$

We then know that the sum of $\tilde{\Phi}_k(v_1)\tilde{\Phi}_k(v_2)e^{-\lambda_k^2 t}$ is $\frac{\sqrt{mn}}{\sqrt{(m+1)(n+1)}(m+n+1)}$ multiplied by the following:

$$\begin{aligned} & \sum_{k=0}^{\infty} -\operatorname{sgn}\left(\cos\left(\sqrt{\lambda_k}\right)\right) e^{-(\lambda_k)^2 t} \\ &= \sum_{k=0}^{\infty} e^{-((2k+1)\pi-\sqrt{\lambda_0})^2 t} - e^{-((2k+2)\pi-\sqrt{\lambda_0})^2 t} - e^{-(2k\pi+\sqrt{\lambda_0})^2 t} + e^{-((2k+1)\pi+\sqrt{\lambda_0})^2 t} \\ &= \sum_{k=0}^{\infty} e^{-(\sqrt{\lambda_0}-(2k+1)\pi)^2 t} - e^{-(\sqrt{\lambda_0}-(2k+2)\pi)^2 t} - e^{-(\sqrt{\lambda_0}+2k\pi)^2 t} + e^{-(\sqrt{\lambda_0}+(2k+1)\pi)^2 t} \\ &= \sum_{k=-\infty}^{\infty} e^{-(\sqrt{\lambda_0}-(2k+1)\pi)^2 t} - e^{-(\sqrt{\lambda_0}-2k\pi)^2 t}. \end{aligned}$$

We can sum $\Phi(v_1)\Phi(v_2)e^{-\lambda^2 t}$ over all of the eigenfunctions to find an explicit expression for the heat kernel at (v_1, v_2) .

$$\begin{aligned} h_t(v_1, v_2) &= \frac{1}{m+n+1} \left(\sum_{k=-\infty}^{\infty} e^{-k^2 \pi^2 t} \cos(k\pi) \right. \\ &\quad \left. + \sqrt{\frac{mn}{(m+1)(n+1)}} \left(\sum_{k=-\infty}^{\infty} e^{-(\sqrt{\lambda_0}-(2k+1)\pi)^2 t} - \sum_{k=-\infty}^{\infty} e^{-(\sqrt{\lambda_0}-2k\pi)^2 t} \right) \right). \end{aligned}$$

We can rewrite this using Jacobi's identity (see Dym [10]). The identity is:

$$\sum_{k=-\infty}^{\infty} e^{-\frac{(x-k)^2}{2s}} = \sqrt{2\pi s} \sum_{k=-\infty}^{\infty} e^{-2\pi^2 k^2 s} e^{2\pi i k x}.$$

Because the left hand side is a real number, we can rewrite the $e^{2\pi i k x}$ on the right hand side to obtain:

$$\sum_{k=-\infty}^{\infty} e^{-\frac{(x-k)^2}{2s}} = \sqrt{2\pi s} \sum_{k=-\infty}^{\infty} e^{-2\pi^2 k^2 s} \cos(2\pi k x).$$

We can apply this identity to the three series. For the first, we use the reverse of the equality with $x = \frac{1}{2}$ and $s = \frac{t}{2}$ in the right hand side. This gives us:

$$\sum_{k=-\infty}^{\infty} e^{-k^2 \pi^2 t} \cos(k\pi) = \frac{1}{\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(1-2k)^2}{4t}}.$$

In the second, let $x = \frac{\sqrt{\lambda_0} + \pi}{2\pi}$ and $s = \frac{1}{8\pi^2 t}$. This gives us $\frac{(x-k)^2}{2s} = \frac{(\sqrt{\lambda_0} + \pi - 2\pi k)^2}{(2\pi)^2 2 \frac{1}{8\pi^2 t}} = (\sqrt{\lambda_0} + (1-2k)\pi)^2 t$. When we put that into the identity, we have:

$$\sum_{k=-\infty}^{\infty} e^{-(\sqrt{\lambda_0} + (1-2k)\pi)^2 t} = \frac{1}{2\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{k^2}{4t}} \cos\left(k\left(\sqrt{\lambda_0} + \pi\right)\right).$$

Note that $\sum_{k=-\infty}^{\infty} e^{-(\sqrt{\lambda_0} - (2k+1)\pi)^2 t} = \sum_{k=-\infty}^{\infty} e^{-(\sqrt{\lambda_0} + (1-2k)\pi)^2 t}$ by reindexing, and so this is a way of rewriting our original sum.

For the third, we use $x = \frac{\sqrt{\lambda_0}}{2\pi}$ and $s = \frac{1}{8\pi^2 t}$ which, by a similar computation, gives us:

$$\sum_{k=-\infty}^{\infty} e^{-(\sqrt{\lambda_0} - 2k\pi)^2 t} = \frac{1}{2\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{k^2}{4t}} \cos\left(k\sqrt{\lambda_0}\right).$$

We can combine these three to rewrite $h_t(v_1, v_2)$ as follows:

$$\begin{aligned} h_t(v_1, v_2) = & \frac{1}{m+n+1} \frac{1}{\sqrt{\pi t}} \left(\sum_{k=-\infty}^{\infty} e^{-\frac{(1-2k)^2}{4t}} \right. \\ & \left. + \sqrt{\frac{mn}{(m+1)(n+1)}} \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{-\frac{k^2}{4t}} \left(\cos\left(k\sqrt{\lambda_0} + k\pi\right) - \cos\left(k\sqrt{\lambda_0}\right) \right) \right). \end{aligned}$$

We can use the angle sum formula for cosines to see that

$$\begin{aligned} \cos\left(k\sqrt{\lambda_0} + k\pi\right) &= \sin\left(k\sqrt{\lambda_0}\right) \sin(k\pi) + \cos\left(k\sqrt{\lambda_0}\right) \cos(k\pi) \\ &= \cos\left(k\sqrt{\lambda_0}\right) \cos(k\pi). \end{aligned}$$

As $\cos(k\pi) = 1$ for k even and -1 for k odd, the term $\cos(k\sqrt{\lambda_0} + k\pi) - \cos(k\sqrt{\lambda_0})$ reduces to 0 for even k and $-2\cos(k\sqrt{\lambda_0})$ for odd k . This allows us to write:

$$\begin{aligned} h_t(v_1, v_2) &= \frac{1}{m+n+1} \frac{1}{\sqrt{\pi t}} \left(\sum_{k=-\infty}^{\infty} e^{-\frac{(2k+1)^2}{4t}} \right. \\ &\quad \left. + \sqrt{\frac{mn}{(m+1)(n+1)}} \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{-\frac{(2k+1)^2}{4t}} (-2) \cos((2k+1)\sqrt{\lambda_0}) \right) \\ &= \frac{1}{m+n+1} \frac{1}{\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(2k+1)^2}{4t}} \left(1 - \frac{\sqrt{mn} \cos((2k+1)\sqrt{\lambda_0})}{\sqrt{(m+1)(n+1)}} \right). \end{aligned}$$

When we consider the limit as t approaches 0, the $k = 0$ and $k = -1$ terms will dominate. By symmetry, these terms are equal. This gives us the approximation:

$$\begin{aligned} h_t(v_1, v_2) &\approx \frac{1}{m+n+1} \frac{2}{\sqrt{\pi t}} e^{-\frac{1}{4t}} \left(1 - \frac{\sqrt{mn} \cos(\sqrt{\lambda_0})}{\sqrt{(m+1)(n+1)}} \right) \\ &= \frac{1}{m+n+1} \frac{2}{\sqrt{\pi t}} e^{-\frac{1}{4t}} \left(1 - \frac{\sqrt{mn} \sqrt{\frac{mn}{(m+1)(n+1)}}}{\sqrt{(m+1)(n+1)}} \right) \\ &= \frac{1}{(m+1)(n+1)} \frac{2}{\sqrt{\pi t}} e^{-\frac{1}{4t}}. \end{aligned}$$

Note that when $m = n = 0$ we have a line; this is the correct asymptotic there. Similarly, when $m = n = 1$, we have a line of length 3. This works out there too.

We can also find the on-diagonal asymptotic at v_1 , one of the central vertices. Here we have

$$\Phi_k^2(v_1) = \frac{2}{m+n+1}$$

and

$$\begin{aligned} \tilde{\Phi}_k^2(v_1) &= \left(\frac{\sqrt{m(m+1)}}{\sqrt{m+n+1}} \right) \sin^2(\sqrt{\lambda_k}) \\ &= \frac{m}{(m+n+1)(n+1)}. \end{aligned}$$

Using the same style of manipulations as before we find that $h_t(v_1, v_1)$ is

$$h_t(v_1, v_1) = \frac{1}{m+n+1} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi^2 t} + \frac{m}{(m+n+1)(n+1)} \sum_{k=-\infty}^{\infty} e^{-(\lambda_0 - k\pi)^2 t}.$$

We can use a Jacobi transform here with $x = 0$, $t = 2s$ in the first sum and $x = \frac{\sqrt{\lambda_0}}{\pi}$ and $t = \frac{1}{2\pi^2 s}$ in the second to obtain:

$$h_t(v_1, v_1) = \frac{1}{(m+n+1)\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} \left(1 + \frac{m}{n+1} \cos(2k\sqrt{\lambda_0}) \right) e^{-\frac{k^2}{t}}.$$

As t approaches zero, the $k \neq 0$ terms also approach zero. The dominant $k = 0$ term is

$$\frac{1}{(m+n+1)\sqrt{\pi t}} \left(1 + \frac{m}{n+1} \right) = \frac{1}{(n+1)\sqrt{\pi t}}.$$

This is the same asymptotic as we have on a single star whose center connects to $n+1$ edges. This is not surprising, since small values of t correspond to the local geometry of a space.

Example 3.2.6. The heat kernel on an interval is messier than that on a line.

Dym [10] calculates it for $\frac{\partial}{\partial t}u - \frac{1}{2}\Delta u = 0$ on the interval $[0, 1]$ to be:

$$\begin{aligned} p_t(x, y) &= 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t/2} \cos(n\pi x) \cos(n\pi y) \\ &= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{(x-y-2n)^2}{2t}} + e^{-\frac{(x+y-2n)^2}{2t}}. \end{aligned}$$

We would like to have the heat kernel for $\frac{\partial}{\partial t}\tilde{u} - \Delta\tilde{u} = 0$ on $[0, L]$. We can find it by noting that if $u(t, x)$ is a solution to $\frac{\partial}{\partial t}u - \frac{1}{2}\Delta u = 0$ on the interval $[0, 1]$, then $\tilde{u}(t, x) = cu(at, bx)$ has derivatives

$$\left(\frac{\partial}{\partial t} \tilde{u} \right) (t, x) = \left(ca \frac{\partial}{\partial t} u \right) (at, bx)$$

and

$$(\Delta \tilde{u})(t, x) = (cb^2 \Delta u)(at, bx) = \left(2cb^2 \frac{\partial}{\partial t} u \right) (at, bx).$$

Then $(\frac{\partial}{\partial t}\tilde{u})(t, x) - (\Delta\tilde{u})(t, x) = 0$ when $a = 2b^2$. We change the interval length by setting $b = \frac{1}{L}$. This gives us $\tilde{p}_t(x, y) = cp_{\frac{2t}{L^2}}(\frac{x}{L}, \frac{y}{L})$. The constant c is used to normalize so that $\int_0^L \tilde{p}_t(x, y)dx = 1$ holds for each $t > 0$. Since this integral is $\int_0^L cp_{\frac{2t}{L^2}}(\frac{x}{L}, \frac{y}{L})dx = \int_0^1 cp_{\frac{2t}{L^2}}(z, \frac{y}{L})Ldz = cL$, we have $c = \frac{1}{L}$.

We can write this as

$$\tilde{p}_t(x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{(x-y-2nL)^2}{4t}} + e^{-\frac{(x+y-2nL)^2}{4t}}.$$

This description allows one to look at terms with n near 0 to find the small time asymptotic.

Note that for $L = 3$, $x = 1$ and $y = 2$ we have:

$$\tilde{p}_t(1, 2) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{(-1-6n)^2}{4t}} + e^{-\frac{(3-6n)^2}{4t}}.$$

When t is small, this behaves like the first $n = 0$ term:

$$\tilde{p}_t(1, 2) \approx \frac{1}{2\sqrt{\pi t}} e^{-\frac{1}{4t}}.$$

This is consistent with the asymptotic for the two star case when $m = n = 1$. Note also that when $L = 1$, $x = 0$, $y = 1$ that this gives $\frac{2}{\sqrt{\pi t}} e^{-\frac{1}{4t}}$ which is consistent with the star with one edge.

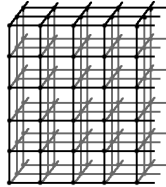


Figure 3.3: A subset of the three dimensional grid.

Example 3.2.7. Complexes can have an underlying group structure. For example, Z^3 , the group consisting of triplets of integers, can be used to create a 3 dimensional

complex by connecting a cube, $[0, 1]^3$, to itself where each pair of triples which differ by one by a line segment. This grid is the 1-skeleton, and the points in the group form a 0-skeleton. We can use the 1-skeleton to create a space that looks like a bunch of empty boxes by filling in the faces formed by loops of four edges. This is the 2-skeleton. If we then fill in the boxes in the 2-skeleton, we'll have a 3-skeleton, which is R^3 . Locally, we've shown that if h_t is the heat kernel on the k -skeleton,

$$\frac{1}{Ct^{k/2}} \leq h_t(x, x) \leq \frac{C}{(\min(t, R_0^2))^{k/2}}$$

for all $x \in X$ and all t such that $0 < t < R_0^2$.

We prove in chapter 5 that the heat kernel on each of these k -skeletons globally behaves like the heat kernel on R^3 .

Example 3.2.8. There are also complexes with underlying group structure whose geometry is not globally Euclidean. Let G be the free group on two elements; this is a group of words formed by letters a and b and their inverses a^{-1} and b^{-1} where the only cancellations are $aa^{-1} = a^{-1}a = 1$ and $bb^{-1} = b^{-1}b = 1$, and a and b don't commute. Let Y be the complex formed by three squares joined into an L shape (see Fig. 3.4). We have a larger structure X which has copies of Y connected to each other via the group G . That is, each copy of Y will be connected to four other copies of Y ; the top of the L connects to the bottom edge of the lower left square of the L, and the right of the L connects to the left edge of the bottom square.

Note that we can not isometrically embed X into R^2 . The stretching of the edges in Fig. 3.4 is to allow you to see distinct edges and vertices. This structure globally it acts like a hyperbolic space, but locally it is Euclidean.

For a small ball with $R < .5$, we have a two dimensional circle (possibly missing

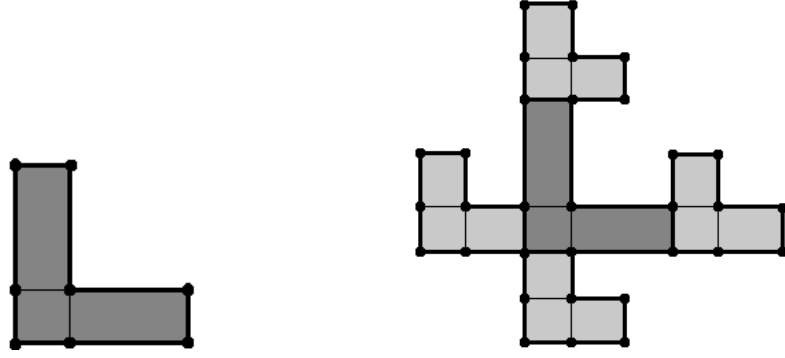


Figure 3.4: Y (left); Y (darker shading) and its four surrounding copies (lighter shading) (right) Each of the edges in these pictures should be interpreted as having length 1.

a wedge) whose volume is $\approx \pi R^2$. For $t < .5$, corollary 3.1.6 tells us:

$$h_t(x, x) \approx \frac{C}{(4\pi t)}.$$

For a given copy of Y , there are four neighbors, each of which has three additional neighbors. For a ball of radius $R > 1$, there will be approximately $1 + 4 + 4(3) + \cdots + 4(3^R) \approx 2(3^{R+1})$ copies of Y . This tells us that for large R , we have exponential volume growth. In particular, this group is nonamenable. We define this in chapter 5 and show that the large time behavior of the heat kernel is

$$\sup_{x \in X} h_t(x, x) \approx C_0 e^{-t/C_1}.$$

Chapter 4

Setup for Groups

A finite product of elements from a set S is called a word. If a word is written $s_1 s_2 \dots s_k$, we say it has length k .

A finitely generated group is a group with a generating set, S , where every element in the group can be written as a finite word using elements of S . Although for a given $g \in G$ it is computationally difficult to determine which word is the smallest one representing g , such a word (or words) exists. If this word has length k , then we write $|g| = k$.

We define the volume of a subset of G to be the number of elements of G contained in that subset. We write $|B_r|$ to denote the volume of a ball of radius r , $B_r := \{g \in G : |g| \leq r\}$. For groups, volume is translation invariant, and so we do not lose any generality by having it centered at the identity.

For a function f which maps elements of a group to the reals, we define the Dirichlet form on $\ell^2(G)$ to be $E(f, f) = \frac{1}{|S|} \sum_{g \in G} \sum_{s \in S} |f(g) - f(gs)|^2$.

Although we'd need to specify directions if we were to define a gradient, we can define an object which behaves like the length of the gradient of f on G . We write this as $|\nabla f(x)| = \sqrt{\frac{1}{|S|} \sum_{s \in S} |f(x) - f(xs)|^2}$. Notationally, this means that $E(f, f) = \sum_{g \in G} |\nabla f(g)|^2$.

Discrete L^p norms restricted to a subset $A \subset G$ are written as

$$\|f\|_{p,A} = \left(\sum_{x \in A} |f(x)|^p \right)^{1/p}. \text{ When } A = G, \text{ we will write } \|f\|_{p,G}.$$

One can show a Poincaré type inequality on a volume doubling finitely generated group. The arguments used in this can be found in [7].

Lemma 4.0.9. *Let G be a finitely generated group with generating set S . For any*

$f : G \rightarrow R$, the following inequality holds on balls B_r :

$$\|f - f_{B_r}\|_{1,B_r} \leq \frac{|B_{2r}|}{|B_r|} 2r \sqrt{|S|} \|\nabla f\|_{1,B_{3r}}.$$

If the group is volume doubling, this is a weak Poincaré inequality on balls for $p = 1$:

$$\|f - f_{B_r}\|_{1,B_r} \leq 2r C_{\text{Doubling}} \sqrt{|S|} \|\nabla f\|_{1,B_{3r}}.$$

Proof. Let G be a finitely generated group with a symmetric set of generators, S . Let B_r be a ball of radius r ; for brevity, we will not explicitly write the center. We can write the norm of f minus its average as follows.

$$\begin{aligned} \|f - f_{B_r}\|_{1,B_r} &= \sum_{x \in B_r} |f(x) - \frac{1}{|B_r|} \sum_{y \in B_r} f(y)| \\ &\leq \frac{1}{|B_r|} \sum_{x \in B_r} \sum_{y \in B_r} |f(x) - f(y)|. \end{aligned}$$

For each $y \in B_r$, there exists a $g \in G$ with $|g| \leq 2r$ such that $y = xg$. We make this substitution and sum over all $g \in G$ with $|g| \leq 2r$.

$$\begin{aligned} \frac{1}{|B_r|} \sum_{x \in B_r} \sum_{y \in B_r} |f(x) - f(y)| &\leq \frac{1}{|B_r|} \sum_{x \in B_r} \sum_{g: |g| \leq 2r} |f(x) - f(xg)| \\ &= \frac{1}{|B_r|} \sum_{g: |g| \leq 2r} \sum_{x \in B_r} |f(x) - f(xg)|. \end{aligned}$$

We will begin with the innermost quantity, and then simplify the sums. We can write $g = s_1 \dots s_k$ as a reduced word with $k \leq 2r$. We rewrite the difference of f at x and xg by splitting the path between them into pieces.

$$|f(x) - f(xg)| \leq \sum_{i=1}^{|g|} |f(xs_1 \dots s_{i-1}) - f(xs_1 \dots s_i)|.$$

We fix g and sum over all $x \in B_r$.

$$\begin{aligned} \sum_{x \in B_r} |f(x) - f(xg)| &\leq \sum_{x \in B_r} \sum_{i=1}^{|g|} |f(xs_1 \dots s_{i-1}) - f(xs_1 \dots s_i)| \\ &= \sum_{i=1}^{|g|} \sum_{x \in B_r} |f(xs_1 \dots s_{i-1}) - f(xs_1 \dots s_i)|. \end{aligned}$$

We can change variables by letting $z = xs_1 \dots s_{i-1}$. Note that $|xs_1 \dots s_{i-1}| < 3r$ as $i \leq 2r$. Then $z \in B_{3r}$.

$$\sum_{i=1}^{|g|} \sum_{x \in B_r} |f(xs_1 \dots s_{i-1}) - f(xs_1 \dots s_i)| \leq \sum_{i=1}^{|g|} \sum_{z \in B_{3r}} |f(z) - f(zs_i)|.$$

Since $s_i \in S$, we can sum over all $s \in S$ instead of the s_i in g . To do this, we must account for the multiplicity of the s_i . We could have at most $|g|$ copies of any generator; $|g| \leq 2r$, and so we will multiply by $2r$.

$$\sum_{i=1}^{|g|} \sum_{z \in B_{3r}} |f(z) - f(zs_i)| \leq 2r \sum_{s \in S} \sum_{z \in B_{3r}} |f(z) - f(zs)|.$$

Jensen's inequality allows us to rewrite this in terms of the gradient.

$$\begin{aligned} 2r \sum_{s \in S} \sum_{z \in B_{3r}} |f(z) - f(zs)| &\leq 2r \sum_{z \in B_{3r}} \sqrt{|S| \frac{1}{|S|} \sum_{s \in S} |f(z) - f(zs)|^2} \\ &= 2r \sum_{z \in B_{3r}} \sqrt{|S|} |\nabla f(z)|. \end{aligned}$$

We'll use this calculation to get the desired inequality. We now have

$$\sum_{x \in B_r} |f(x) - f(xg)| \leq 2r \sum_{z \in B_{3r}} \sqrt{|S|} |\nabla f(z)|.$$

Dividing by $|B_r|$ and summing over the g gives us

$$\begin{aligned} \frac{1}{|B_r|} \sum_{g: |g| \leq 2r} \sum_{x \in B_r} |f(x) - f(xg)| &\leq \frac{1}{|B_r|} \sum_{g: |g| \leq 2r} \sum_{z \in B_{3r}} 2r \sqrt{|S|} |\nabla f(z)| \\ &= \frac{|B_{2r}|}{|B_r|} 2r \sqrt{|S|} \sum_{z \in B_{3r}} |\nabla f(z)|. \end{aligned}$$

This reduces to

$$\|f - f_{B_r}\|_{1, B_r} \leq \frac{|B_{2r}|}{|B_r|} 2r \sqrt{|S|} \|\nabla f\|_{1, B_{3r}}.$$

Note that in general, $\frac{|B_{2r}|}{|B_r|}$ will depend on the radius, r . If the group is volume doubling, this gives us a weak Poincaré inequality.

$$\|f - f_{B_r}\|_{1, B_r} \leq C_{\text{Doubling}} 2r \sqrt{|S|} \|\nabla f\|_{1, B_{3r}}.$$

□

4.1 Comparing distances in X and G

Let X be a complex, and G be a finitely generated group of isomorphisms on the complex such that $X/G = Y$ is an admissible complex consisting of a finite number of polytopes.

One example of this type of complex is a Cayley graph; this is the graph where each vertex corresponds to a group element, and two vertices are connected by an edge if they differ by an element of the generating set. In this case, Y is the unit interval.

We would like to be able to compare functions defined on the group, G , with functions defined on the complex, X . To do this, we will look at ways to transfer a function defined on G to a function defined on X that roughly preserves the norm of both the function and its energy form. We seek to do the reverse as well. We will use a technique that originated with Kanai [19] and additionally was used by Coulhon and Saloff-Coste [8].

We also want a way of changing from real valued functions which take values in X to ones that take values in G . We will do this by taking a copy of Y and splitting it into many smaller pieces. Given $\delta \leq \text{diam}(Y)$, we can find a finite covering of Y by balls of radius δ such that balls of radius $\delta/2$ are disjoint in Y . As Y is a finite polytopal complex, volume doubling on Y implies that at most a finite number of balls of radius δ will overlap. X can be written by taking a copy of Y for each element of G , and so this cover can be expanded to a cover of X . Note that once we have a copy of the cover of Y for each element of G , balls of radius $\delta/2$ in this larger cover may overlap. For X , the overlap is also finite; call the number of overlapping balls $C_{N,S}$.

Call the centers of the balls covering Y $\{\gamma_i\}_{i=1}^N$ and the balls covering X

$\{g\gamma_i\}_{i=1..N; g \in G}$. Note that each $x \in X$ is within δ of at least one of the $g\gamma_i$. As we are frequently switching between X and G , we will use B_X for balls in X and B_G for balls in G .

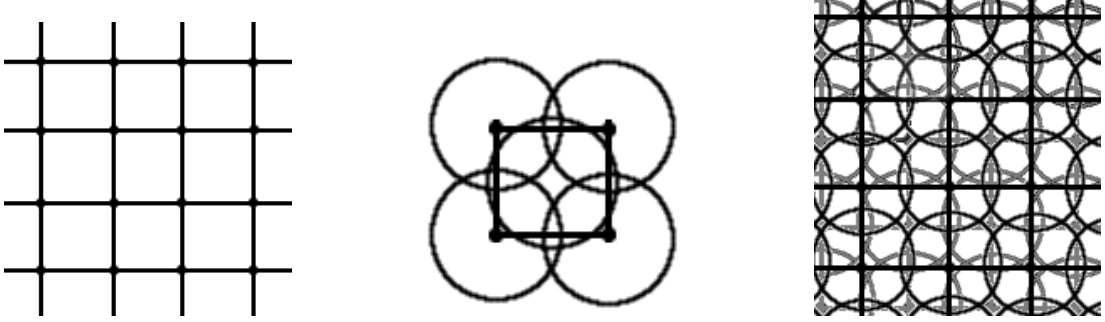


Figure 4.1: X split into copies of Y (left); a copy of Y covered by 5 balls of radius .6 (center); a cover for Y shifted by five copies of G cover everything – the black lines represent four copies that overlap exactly; the fifth copy is gray(right).

Example 4.1.1. Let $X = \mathbb{R}^2$, $G = \mathbb{Z}^2$ and $Y = [0, 1]^2$. For $\delta = .6$, balls of radius $\delta/2 = .3$ centered at the corners of Y are disjoint, but not all points in the plane are covered. We can introduce another copy of G that's shifted by $(.5, .5)$. All points in Y are covered by some ball of radius delta, but the balls of radius .3 will not overlap. We can check this by comparing the distance along the diagonal from $(0, 0)$ to $(1, 1)$ with the length covered by the radii along the same diagonal. We have $d((0, 0), (1, 1)) = \sqrt{2} \approx 1.4$. For the balls, there's the radius of the one centered at $(0, 0)$, the one centered at $(1, 1)$ and the diameter of the one centered at $(.5, .5)$. This sums to $4\delta = 1.2$.

The number of overlapping balls in X is $C_{N,S} = 10$. This happens at $(0, .5)$ where there are four balls centered at $(0, 0)$, one centered at $(.5, .5)$, one centered at $(- .5, .5)$, and four centered at $(0, 1)$. In this example, $\gamma_1 = (0, 0)$, $\gamma_2 = (0, 1)$, $\gamma_3 = (1, 0)$, $\gamma_4 = (1, 1)$, and $\gamma_5 = (.5, .5)$.

We define

$$\text{group } f(g, i) = \frac{1}{\mu(B_X(g\gamma_i, \delta))} \int_{B_X(g\gamma_i, \delta)} f(x) dx = \int_{B_X(g\gamma_i, \delta)} f(x) dx.$$

We can view group f as a collection of N functions defined on G . For any fixed i , we can treat group $f(\cdot, i)$ as a function on the group, and so the norm of group f can be found by summing these over i .

It's important to note that the sets $\{g : g \in B_X(r) \cap G\}$ and $B_G(r)$ are potentially different. We can describe this by comparing distances with some explicit constants.

Lemma 4.1.2. *We can compare distances in G and X in the following manner.*

There exist constants C_{XG} and C_0 so that for any $g, h \in G$ we have:

$$\frac{1}{C_0} d_X(g, h) \leq d_G(g, h) \leq C_{XG} d_X(g, h).$$

This tells us that balls centered at points in G compare as:

$$G \cap B_X\left(\frac{r}{C_{XG}}\right) \subset B_G(r) \subset G \cap B_X(C_0 r).$$

Here, $C_0 = \max_{v_1, v_2 \in Y^{(0)}} d_{Y^{(1)}}(v_1, v_2)$ and

$$C_{XG} = \frac{1}{\min_{v_1, v_2 \in Y^{(0)}} d_{X^{(1)}}(v_1, v_2)} \prod_{i=1}^n \max \left(\sqrt{\frac{2}{1-\cos(\alpha)}}, \frac{\max_{y_1, y_2 \in Y^{(i-1)}} d_{X^{(i-1)}}(y_1, y_2)}{\min_{v_1, v_2 \in Y^{(0)}} d_{X^{(1)}}(v_1, v_2)} \right) \text{ where}$$

α is the smallest interior angle in X .

Proof. We'll start with the easy direction. Any path in $X^{(1)}$ is also a path in X , and so $d_X(g, h) \leq d_{X^{(1)}}(g, h)$. To compare this with distances in G , which count numbers of points in paths, we use lengths of edges between them.

$$d_{X^{(1)}}(g, h) \leq \max_{v_1, v_2 \in Y^{(0)}} d_{Y^{(1)}}(v_1, v_2) d_G(g, h).$$

Set $C_0 = \max_{v_1, v_2 \in Y^{(0)}} d_{Y^{(1)}}(v_1, v_2)$. Since whenever $C_0 d_G(g, h) \leq r$ we also have $d_X(g, h) \leq r$, we know that any point in $B_G(\frac{r}{C_0})$ is also in $B_X(r)$. This tells us that $B_G(r) \subset G \cap B_X(C_0 r)$.

We can use the fact that X can be subdivided into copies of Y to relate distances in the other direction as well. We will compare distances in the different skeletons of X . In order to simplify notation, $d_k(x, y)$ will refer to the distance between x and y when we restrict to paths in $X^{(k)}$.

Let $g, h \in G$ be given. Then there is a shortest path in X between them. If there are multiple such paths, pick one. Label it $\{x_0 = g, x_1, x_2, \dots, x_k = h\}$ where $x_i \in X^{(n-1)}$ for $i = 1..k-1$, and x_i, x_{i+1} are both in the boundary of the same maximal polytope, although they are on different faces. In particular, both are in $X^{(n-1)}$. Then we know that the length of this path is $\sum_{i=1}^k d_n(x_{i-1}, x_i)$.

We will compare $d_n(x_{i-1}, x_i)$ with $d_{n-1}(x_{i-1}, x_i)$, and use this to relate the distances between skeletons who differ by 1 dimension. This will allow us to work our way from n dimensions down to $X^{(0)}$. Then we can compare $X^{(0)}$ with G .

Look at x_i and x_{i+1} which are in $(n-1)$ dimensional faces F_i and F_{i+1} . Either $F_i \cap F_{i+1}$ is nonempty and so they share a lower dimensional point, or else it is empty and they do not.

If they do not share a point, then:

$$\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2) \leq d_n(x_i, x_{i+1}).$$

We get this bound because the diameter of any subpolyhedra of Y must be bounded below by the length of the smallest edge of that polyhedra.

We can also bound the $(n-1)$ distance:

$$d_{n-1}(x_i, x_{i+1}) \leq \max_{y_1, y_2 \in Y^{(n-1)}} d_{n-1}(y_1, y_2).$$

Putting this together, we get:

$$d_{n-1}(x_i, x_{i+1}) \leq \frac{\max_{y_1, y_2 \in Y^{(n-1)}} d_{n-1}(y_1, y_2)}{\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2)} d_n(x_i, x_{i+1}).$$

Otherwise, if F_i and F_{i+1} intersect in a lower dimensional face, we will call v the point on the intersection which minimizes $d_{n-1}(v, x_i) + d_{n-1}(v, x_{i+1})$. These three points form a triangle with angle $x_i v x_{i+1} = \theta \geq \alpha$, where α is the smallest interior angle in Y as well as in X . Note that this angle is bounded because Y is made up of a finite number of polytopes. We would like to determine a relationship between $d_n(x_i, x_{i+1})$ and $\inf_{v \in F_i \cap F_{i+1}} d_{n-1}(v, x_i) + d_{n-1}(v, x_{i+1})$.

To find this, we will use a simple derivation. For positive numbers a and b we have

$$\begin{aligned} (a - b)^2 &\geq -\cos(\alpha)(a - b)^2 \\ a^2 + b^2 - 2ab\cos(\alpha) &\geq 2ab - a^2\cos(\alpha) - b^2\cos(\alpha) \\ 2a^2 + 2b^2 - 4ab\cos(\alpha) &\geq a^2 + b^2 + 2ab - 2ab\cos(\alpha) - a^2\cos(\alpha) - b^2\cos(\alpha) \\ a^2 + b^2 - 2ab\cos(\alpha) &\geq (a + b)^2 \left(\frac{1 - \cos(\alpha)}{2} \right). \end{aligned}$$

This is helpful because when we apply the law of cosines to the triangle we have:

$$d_n^2(x_i, x_{i+1}) = d_{n-1}^2(x_i, v) + d_{n-1}^2(v, x_{i+1}) - 2d_{n-1}(x_i, v)d_{n-1}(v, x_{i+1})\cos(\theta).$$

We can form an inequality by replacing $\cos(\theta)$ with the larger $\cos(\alpha)$:

$$d_n^2(x_i, x_{i+1}) \geq d_{n-1}^2(x_i, v) + d_{n-1}^2(v, x_{i+1}) - 2d_{n-1}(x_i, v)d_{n-1}(v, x_{i+1})\cos(\alpha).$$

Then we can apply our fact with $a = d_{n-1}(x_i, v)$ and $b = d_{n-1}(v, x_{i+1})$.

$$d_n^2(x_i, x_{i+1}) \geq (d_{n-1}(x_i, v) + d_{n-1}(v, x_{i+1}))^2 \left(\frac{1 - \cos(\alpha)}{2} \right).$$

This leads us to the conclusion that

$$\begin{aligned} d_n(x_i, x_{i+1}) &\geq (d_{n-1}(x_i, v) + d_{n-1}(v, x_{i+1})) \sqrt{\frac{1 - \cos(\alpha)}{2}} \\ &\geq d_{n-1}(x_i, x_{i+1}) \sqrt{\frac{1 - \cos(\alpha)}{2}}. \end{aligned}$$

When we combine the cases where faces intersect with the case where they do not, we get the following inequality:

$$d_{n-1}(x_i, x_{i+1}) \leq \max \left(\sqrt{\frac{2}{1 - \cos(\alpha)}}, \frac{\max_{y_1, y_2 \in Y^{(n-1)}} d_{n-1}(y_1, y_2)}{\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2)} \right) d_n(x_i, x_{i+1}).$$

We can sum and use the fact that we had a distance minimizing path in $X^{(n)}$ to get

$$d_{n-1}(g, h) \leq \max \left(\sqrt{\frac{2}{1 - \cos(\alpha)}}, \frac{\max_{y_1, y_2 \in Y^{(n-1)}} d_{n-1}(y_1, y_2)}{\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2)} \right) d_n(g, h).$$

We can repeat this argument for the lower dimensions (down to dimension 1) to get:

$$d_1(g, h) \leq \prod_{i=1}^n \max \left(\sqrt{\frac{2}{1 - \cos(\alpha)}}, \frac{\max_{y_1, y_2 \in Y^{(i-1)}} d_{i-1}(y_1, y_2)}{\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2)} \right) d_{X^{(n)}}(g, h).$$

To compare with the distance in G , we see that

$$\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2) d_G(g, h) \leq d_1(g, h).$$

We will define C_{XG} to be

$$C_{XG} = \frac{1}{\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2)} \prod_{i=1}^n \max \left(\sqrt{\frac{2}{1 - \cos(\alpha)}}, \frac{\max_{y_1, y_2 \in Y^{(i-1)}} d_{i-1}(y_1, y_2)}{\min_{v_1, v_2 \in Y^{(0)}} d_1(v_1, v_2)} \right).$$

This gives us the inequality:

$$d_G(g, h) \leq C_{XG} d_X(g, h).$$

The inequality implies the containment $G \cap B_X(\frac{r}{C_{XG}}) \subset B_G(r)$ by the argument from the start of this proof. \square

4.2 Comparing functions on X with corresponding ones on G

We can compare the norm of f with the norm of group f , as well as the norm of ∇f with that of its analogue. Note that given a radius, R , Corollary 2.2.9 tells us

that we have a uniform Poincaré inequality for f on balls of radius at most R . This will be helpful for our comparison. In particular, we will use this where C_P is the constant associated to the Poincaré inequality for balls of radius up to $3 \text{diam}(Y)$. Note that if we took $\delta = \text{diam}(Y)$, we could cover Y with exactly one ball.

Lemma 4.2.1. *Let $B_X(r) := B_X(g', r)$ be a ball in X centered at $g' \in G$. For any $c \in R$, we can compare $f : X \rightarrow R$ with group $f : G^N \rightarrow R$ in the following manner:*

$$\|f - c\|_{p, B_X(r)}^p \leq C \left(\delta^p \|\nabla f\|_{p, B_X(r+2\delta)}^p + \|\text{group } f - c\|_{p, B_G(C_{XG}(r+\delta+\text{diam}(Y)))}^p \right).$$

When $r = \infty$, this says that:

$$\|f - c\|_{p, X}^p \leq C \left(\delta^p \|\nabla f\|_{p, X}^p + \|\text{group } f - c\|_{p, G}^p \right).$$

The constant C depends on X , p , and δ .

Proof. We begin by rewriting the norm using the fact that balls of radius $r + \delta$ centered at $g\gamma_i$ form a cover.

$$\begin{aligned} \|f - c\|_{p, B_X(r)}^p &= \int_{B_X(r)} |f(x) - c|^p dx \\ &\leq \sum_i \sum_{g\gamma_i \in B_X(r+\delta)} \int_{B_X(g\gamma_i, \delta)} |f(x) - c|^p dx. \end{aligned}$$

We'll use the fact that $|f(x) - c|^p \leq 2^p |f(x) - \text{group } f(g, i)|^p + 2^p |\text{group } f(g, i) - c|^p$ to split this into two pieces. In the first piece, we can simplify using the local Poincaré inequality in X .

$$\begin{aligned} \sum_i \sum_{g\gamma_i \in B_X(r+\delta)} 2^p \int_{B_X(g\gamma_i, \delta)} |f(x) - \text{group } f(g, i)|^p dx \\ \leq \sum_i \sum_{g\gamma_i \in B_X(r+\delta)} 2^p \delta^p C_P \int_{B_X(g\gamma_i, \delta)} |\nabla f(x)|^p dx \\ \leq 2^p \delta^p C_P C_{N,S} \int_{B_X(r+2\delta)} |\nabla f(x)|^p dx. \end{aligned}$$

In the second, we first note that there is no x dependence in the integrand. We integrate to get the volume of the ball. This will be dominated by the largest such volume.

$$\begin{aligned}
& \sum_i \sum_{g\gamma_i \in B_X(r+\delta)} 2^p \int_{B_X(g\gamma_i, \delta)} |\text{group } f(g, i) - c|^p dx \\
&= \sum_i \sum_{g\gamma_i \in B_X(r+\delta)} 2^p \mu(B_X(g\gamma_i, \delta)) |\text{group } f(g, i) - c|^p \\
&\leq 2^p \left(\max_{g\gamma_i \in B_X(r+\delta)} \mu(B_X(g\gamma_i, \delta)) \right) \sum_i \sum_{g\gamma_i \in B_X(r+\delta)} |\text{group } f(g, i) - c|^p
\end{aligned}$$

This gives us the definition of the p norm in X . We switch to the norm in G using the distance comparisons from Lemma 4.1.2.

$$\begin{aligned}
& \dots = 2^p \max_{g\gamma_i \in B_X(r+\delta)} \mu(B_X(g\gamma_i, \delta)) \|\text{group } f - c\|_{p, B_X(r+\delta)}^p \\
&\leq 2^p \max_{g\gamma_i \in B_X(r+\delta)} \mu(B_X(g\gamma_i, \delta)) \|\text{group } f - c\|_{p, B_G(C_{XG}(r+\delta+\text{diam}(Y)))}^p.
\end{aligned}$$

When we put these together we have for some constant C :

$$\|f - c\|_{p, B_X(r)}^p \leq C \left(\delta^p \|\nabla f\|_{p, B_X(r+2\delta)}^p + \|\text{group } f - c\|_{p, B_G(C_{XG}(r+\delta+\text{diam}(Y)))}^p \right).$$

Note that the uniformity of X tells us that $\mu(B_X(g\gamma_i, \delta))$ can be bounded by a constant. In particular, we use the fact that $\mu(B_X(g\gamma_i, \delta)) = \mu(B_X(h\gamma_i, \delta))$ for any $g, h \in G$. \square

We can also bound the gradients of f and $\text{group } f$ in their respective norms.

Lemma 4.2.2. *Let $f \in \text{Lip}(X)$ and $B_G(r)$ be given. For $1 \leq p < \infty$, we have*

$$\|\nabla \text{group } f\|_{p, B_G(r)}^p \leq C(\delta) \|\nabla f\|_{p, B_X(C_0 r + 2 \text{diam}(Y))}^p.$$

When $r = \infty$, this is:

$$\|\nabla \text{group } f\|_{p, G}^p \leq C(\delta) \|\nabla f\|_{p, X}^p.$$

$$C(\delta) = \max_{\gamma_i} \frac{\mu(B_X(g\gamma_i, R))}{\mu(B_X(g\gamma_i, \delta))^2} N \max_{x \in X} \#\{B_X(g, R) | x \in B_X(g, R)\} C_P R^p.$$

Note that this constant depends on X , p , N , and δ .

Proof. Here, R is a large enough radius so that for any g and γ_i both $B_X(g\gamma_i, \delta)$ and $B_X(gs\gamma_i, \delta)$ are covered by $B_X(g\gamma_i, R)$. Note that $R = \text{diam}(Y) + \delta$ will work, but to remove the dependence on δ , we can take $R = 2 \text{diam}(Y)$.

We start by explicitly writing out the gradient and then moving the $p/2$ into the integral via Jensen.

$$\begin{aligned} & \|\nabla \text{group } f\|_{p, B_G(r)}^p \\ &= \sum_i \sum_{g \in B_G(r)} \left(\frac{1}{|S|} \sum_{s \in S} |\text{group } f(g, i) - \text{group } f(gs, i)|^2 \right)^{p/2} \\ &\leq \sum_i \sum_{g \in B_G(r)} \frac{1}{|S|} \sum_{s \in S} |\text{group } f(g, i) - \text{group } f(gs, i)|^p \\ &= \sum_i \sum_{g \in B_G(r)} \frac{1}{|S|} \sum_{s \in S} \left| \int_{B_X(g\gamma_i, \delta)} f(x) dx - \int_{B_X(gs\gamma_i, \delta)} f(y) dy \right|^p. \end{aligned}$$

We apply Jensen again; this time to the absolute value.

$$\dots \leq \sum_i \sum_{g \in B_G(r)} \frac{1}{|S|} \sum_{s \in S} \int_{B_X(g\gamma_i, \delta)} \int_{B_X(gs\gamma_i, \delta)} |f(x) - f(y)|^p dx dy.$$

The regularity of the space and the cover tell us $\mu(B_X(g\gamma_i, \delta)) = \mu(B_X(gs\gamma_i, \delta))$.

$$\dots = \sum_i \sum_{g \in B_G(r)} \frac{1}{|S|} \sum_{s \in S} \int_{B_X(g\gamma_i, \delta)} \int_{B_X(gs\gamma_i, \delta)} |f(x) - f(y)|^p \frac{dx dy}{\mu(B_X(g\gamma_i, \delta))^2}.$$

We expand the sets we are integrating over to $B_X(g\gamma_i, R)$. This larger set contains both $B_X(g\gamma_i, \delta)$ and $B_X(gs\gamma_i, \delta)$ by construction. We then rewrite the sum over S , and change one integral to an average integral.

$$\begin{aligned} \dots &\leq \frac{1}{|S|} \sum_i \sum_{g \in B_G(r)} \sum_{s \in S} \frac{1}{\mu(B_X(g\gamma_i, \delta))^2} \int_{B_X(g\gamma_i, R)} \int_{B_X(g\gamma_i, R)} |f(x) - f(y)|^p dx dy \\ &= \frac{1}{|S|} \sum_i \sum_{g \in B_G(r)} |S| \frac{\mu(B_X(g\gamma_i, \delta))}{\mu(B_X(g\gamma_i, \delta))^2} \int_{B_X(g\gamma_i, R)} \int_{B_X(g\gamma_i, R)} |f(x) - f(y)|^p dx dy. \end{aligned}$$

We now apply a local p Poincaré inequality on X to f . The constant for this is C_P .

$$\dots \leq \sum_i \sum_{g \in B_G(r)} C_P R^p \frac{\mu(B_X(g\gamma_i, R))}{\mu(B_X(g\gamma_i, \delta))^2} \int_{B_X(g\gamma_i, R)} |\nabla f(x)|^p dx.$$

We combine the sums and integral into a single integral. All of the $B_X(g\gamma_i, R)$ for $g \in B_G(r)$ are contained in $B_X(C_0 r + R)$ by our distance comparison between G and X . We multiply this integral by the number of overlapping balls in our sum. C_M is N times the maximum number of balls $B_X(g\gamma_i, R)$ which overlap at a point in X .

$$\begin{aligned} \dots &\leq \left(\max_{\gamma_i} \frac{\mu(B_X(g\gamma_i, R))}{\mu(B_X(g\gamma_i, \delta))^2} \right) C_M C_P R^p \int_{B_X(C_0 r + R)} |\nabla f(x)|^p dx \\ &= C(\delta) \|\nabla f\|_{p, B_X(C_0 r + R)}^p. \end{aligned}$$

□

4.3 Poincaré inequalities on X with underlying group structure

The bounds in the previous section can be used to transfer inequalities between X and G . We can combine them with the weak Poincaré inequality on G to get an inequality on X .

Theorem 4.3.1. *Let X , a volume doubling Euclidean complex and G , a finitely generated group with $X/G = Y$, a finite admissible polytopal complex be given. X admits a Poincaré inequality with uniform constant at all scales. Let $f \in \text{Lip}(X)$. For $1 \leq p < \infty$, we have:*

$$\inf_c \|f - c\|_{p, B_X(r)} \leq Cr \|\nabla f\|_{p, B_X(r)}.$$

Note that this implies:

$$\|f - f_{B_X(r)}\|_{p, B_X(r)} \leq 2Cr \|\nabla f\|_{p, B_X(r)}.$$

Here the balls can be centered at any point in X .

Proof. Note that we chose C_P so that the Poincaré inequality holds for balls of radius up to $3 \operatorname{diam}(Y)$. We need to show that it also holds for balls of radius greater than $3 \operatorname{diam}(Y)$. Let $r \geq 3 \operatorname{diam}(Y)$ be given. To start, we will assume that the center of $B_X(r)$ is in G . Pick $\delta = \operatorname{diam}(Y)$; this will force $N = 1$. This will allow us to split things up in such a way that we can use the weak Poincaré inequality on G . If we had multiple copies of G , we wouldn't necessarily have the same average on each of them. By choosing a value of c , we obtain something at least as large as the infimum:

$$\inf_c \|f - c\|_{1, B_X(r)} \leq \|f - \operatorname{group} f_{B_G(2C_{XG}r)}\|_{1, B_X(r)}.$$

Then we can use our first bound to get

$$\begin{aligned} & \|f - \operatorname{group} f_{B_G(2C_{XG}r)}\|_{1, B_X(r)} \\ & \leq C \left(\delta \|\nabla f\|_{1, B_X(r+\delta)} + \|\operatorname{group} f - \operatorname{group} f_{B_G(2C_{XG}r)}\|_{1, B_G(C_{XG}(r+\delta+\operatorname{diam}(Y)))} \right) \\ & \leq C \left(\delta \|\nabla f\|_{1, B_X(1.5r)} + \|\operatorname{group} f - \operatorname{group} f_{B_G(2C_{XG}r)}\|_{1, B_G(2C_{XG}r)} \right). \end{aligned}$$

Happily, we can apply the weak Poincaré inequality on groups (Lemma 4.0.9) to the second term:

$$\|\operatorname{group} f - \operatorname{group} f_{B_G(2C_{XG}r)}\|_{1, B_G(2C_{XG}r)} \leq 3r \sqrt{|S|} \|\nabla \operatorname{group} f\|_{1, B_G(6C_{XG}r)}.$$

Then, we can use the bound we have on the gradients to get an inequality on ∇f .

Setting $R = 2 \operatorname{diam}(Y) < r$ tells us that $B_X((6r + R)C_0C_{XG}) \subset B_X(7C_0C_{XG}r)$.

$$\|\nabla \operatorname{group} f\|_{1, B_G(6C_{XG}r)} \leq C(\delta) \|\nabla f\|_{1, B_X(7C_0C_{XG}r)}.$$

Combining these, we have:

$$\begin{aligned} \inf_c \|f - c\|_{1, B_X(r)} &\leq C \left(\delta \|\nabla f\|_{1, B_X(1.5r)} + 3r \sqrt{|S|} C(\delta) \|\nabla f\|_{1, B_X(7C_0 C_{XG} r)} \right) \\ &\leq C_0 r \|\nabla f\|_{1, B_X(7C_0 C_{XG} r)}. \end{aligned}$$

As in lemma 2.1.9, we have:

$$\|f - f_{B_X(r)}\|_{1, B_X(r)} \leq 2 \inf_c \|f - c\|_{1, B_X(r)}.$$

If the center, x , were not in G , there is some $g' \in G$ such that the center is within $\text{diam}(Y)$ of g' . That is, $d_X(x, g') \leq \text{diam}(Y)$. By inclusions of balls, we know that:

$$\inf_c \|f - c\|_{1, B_X(x, r)} \leq \inf_c \|f - c\|_{1, B_X(g', r + \text{diam}(Y))}.$$

As $r + \text{diam}(Y) \leq 1.5r$ and $B_X(g', 7C_0 C_{XG} 1.5r) \subset B_X(x, 12C_0 C_{XG} r)$, we can switch centers by increasing the radius:

$$\|\nabla f\|_{1, B_X(g', 7C_0 C_{XG} 1.5r)} \leq \|\nabla f\|_{1, B_X(x, 12C_0 C_{XG} r)}.$$

This tells us that any complex X with the underlying group structure admits a weak $p = 1$ Poincaré inequality.

X is volume doubling, and so this weak inequality can be turned into a strong p inequality via repeated application of a Whitney cover, using Corollary 2.2.9. \square

In [32] Varopolous showed that groups with polynomial growth of degree d have on diagonal behavior $p_{2n}(e, e) \approx n^{-d/2}$. We show that a similar result holds for volume doubling complexes with underlying group structure.

Theorem 4.3.2. *Assume X is a volume doubling Euclidean complex and G is a finitely generated group with $X/G = Y$, where Y is a finite admissible polytopal complex.*

Then X satisfies the on diagonal heat kernel estimates:

$$\frac{1}{C\mu(B(x, \sqrt{t}))} \leq h_t(x, x) \leq \frac{C}{\mu(B(x, \sqrt{t}))}$$

X also satisfies the off diagonal heat kernel lower bound:

$$\frac{1}{C\mu(B(x, \sqrt{t}))} \exp\left(-C\frac{d_X^2(x, y)}{t}\right) \leq h_t(x, y),$$

as well as the upper bound:

$$h_t(x, y) \leq \frac{C}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \exp\left(-\frac{d_X^2(x, y)}{4t}\right) \left(1 + \frac{d_X^2(x, y)}{t}\right)^{n/2}.$$

Proof. To get the heat kernel bounds, apply Sturm [29] theorems 3.1.1 and 3.1.3, noting that we've satisfied both volume doubling and a Poincaré inequality at all scales uniformly. \square

4.4 Mapping functions on G to X

Now we will look at how to take functions on G to smooth versions on X . Let f be a function mapping G to the reals. We'll look at a partition of unity on the complex, X , which is created by translating a smooth function χ by $g \in G$. Then $\sum_{g \in G} \chi_g(x) = 1$. We require the following:

- $\chi_g(x) = 1$ if $d_X(x, g) \leq \frac{1}{4}$
- $\chi_g(x) = 0$ if $d_X(x, g) \geq C_{sup}$
- $|\nabla \chi_g(x)| \leq C_g$

We know that $|\{g \in G : d_X(g, e) \leq C_{sup}\}|$ is finite; when Y is nice and $C_{sup} = 1$ this will be $|S|$. We also know that for any $x \in X$, $|\{g \in G : \chi_g(x) \neq 0\}|$ is finite. In particular, there is a uniform bound, C_{Over} .

This allows us to define a nice smooth function, $\text{comp } f(x)$, mapping X to the reals:

$$\text{comp } f(x) = \sum_{g \in G} f(g) \chi_g(x).$$

Its L^p norm is comparable to that of f .

Theorem 4.4.1. *Let $f : G \rightarrow R$ be given. If we limit ourselves to a ball, $B_G(r)$, with radius at least 1, we can compare L^p norms in the following way:*

$$C_1 \|f - c\|_{p, B_G(\frac{r-.25}{C_0})} \leq \|\text{comp } f - c\|_{p, B_X(r)} \leq C_2 \|f - c\|_{p, B_G(C_{XG}(r+C_{sup}))}$$

This holds for any $c \in R$. Note that when $r = \infty$ and $c = 0$ we have a nice bound on the norms:

$$C_1 \|f\|_{p, G} \leq \|\text{comp } f\|_{p, X} \leq C_2 \|f\|_{p, G}.$$

For both of these inequalities, $C_1 = \mu(B_X(e, \frac{1}{4}))^{\frac{1}{p}}$ and $C_2 = C_{Over}^{(p-1)/p} \|\chi_e\|_{p, X}$.

Proof. If we limit ourselves to a ball, $B_G(r)$, with radius at least 1, we have a comparison. We first write out the definition of the norm, and then we use the fact that for every x , $\sum_{g \in G} \chi_g(x) = 1$.

$$\begin{aligned} \|\text{comp } f - c\|_{p, B_X(r)} &= \left(\int_{B_X(r)} \left| \sum_{g \in G} f(g) \chi_g(x) - c \right|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int_{B_X(r)} \left| \sum_{g \in G} (f(g) - c) \chi_g(x) \right|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

For each x , at most C_{Over} of the $\chi_g(x)$ are nonzero. This allows us to apply a discrete version of Jensen to move the exponent into the sum.

$$\dots \leq \left(\int_{B_X(r)} C_{Over}^{p-1} \sum_{g \in G} |f(g) - c|^p \chi_g(x)^p dx \right)^{\frac{1}{p}}.$$

The only g with a nonzero $\chi_g(x)$ will be those within X distance C_{sup} of a point in $B_X(r)$. We can integrate over $g \in G \cap B_X(r + C_{sup})$, and switch the finite integral and sum.

$$\dots \leq C_{Over}^{(p-1)/p} \left(\sum_{g \in G \cap B_X(r + C_{sup})} |f(g) - c|^p \int_{B_X(r)} \chi_g(x)^p dx \right)^{\frac{1}{p}}.$$

The quantity $\int_{B_X(r)} \chi_g(x)^p$ will be bounded above by $\int_X \chi_g(x)^p = \int_X \chi_e(x)^p$.

$$\dots \leq C_{Over}^{(p-1)/p} \|\chi_e\|_{p,X} \left(\sum_{g \in G \cap B_X(r + C_{sup})} |f(g) - c|^p \right)^{\frac{1}{p}}.$$

We then use the distance comparisons from lemma 4.1.2 to get a norm with respect to distance in G .

$$\dots \leq C_{Over}^{(p-1)/p} \|\chi_e\|_{p,X} \|f - c\|_{p, B_G(C_{XG}(r + C_{sup}))}.$$

Now we will show the other inequality. By definition, we can write the norm in G as:

$$\|f - c\|_{p, B_G(r)} = \left(\sum_{g \in B_G(r)} |f(g) - c|^p \right)^{\frac{1}{p}}.$$

We introduce χ_g by noting $\chi_g(x) = 1$ for x in $B_X(g, \frac{1}{4})$, and integrating over this set. Due to the regularity of X , $\mu(B_X(g, \frac{1}{4}))$ does not depend on g , and so we write it as $\mu(B_X(e, \frac{1}{4}))$.

$$\dots = \left(\sum_{g \in B_G(r)} \frac{1}{\mu(B_X(e, \frac{1}{4}))} \left(\int_{B_X(g, \frac{1}{4})} |f(g) - c|^p \chi_g(x) dx \right) \right)^{\frac{1}{p}}.$$

We now will switch the integral and the sum. We are integrating only over x in balls centered at points in $B_G(r)$ of radius $1/4$. This set can be written $\cup_{h \in B_G(r)} B_X(h, \frac{1}{4})$.

$$\dots \leq \left(\frac{1}{\mu(B_X(e, \frac{1}{4}))} \int_{\cup_{h \in B_G(r)} B_X(h, \frac{1}{4})} \sum_{g \in G} |f(g) - c|^p \chi_g(x) dx \right)^{\frac{1}{p}}.$$

For x in $\cup_{h \in B_G(r)} B_X(h, \frac{1}{4})$, $\chi_g(x) = 1$ for exactly one $g \in G$, and it is zero otherwise.

This tells us $\sum_{g \in G} |f(g) - c|^p \chi_g(x) = |\sum_{g \in G} (f(g) - c) \chi_g(x)|^p$. We then can write the above as

$$\dots = \left(\frac{1}{\mu(B_X(e, \frac{1}{4}))} \int_{\cup_{h \in B_G(r)} B_X(h, \frac{1}{4})} \left| \sum_{g \in G} (f(g) - c) \chi_g(x) \right|^p dx \right)^{\frac{1}{p}}.$$

We use the distance comparisons from Lemma 4.1.2 to see that $\cup_{h \in B_G(r)} B_X(h, \frac{1}{4}) \subset B_X(C_0 r + .25)$.

$$\dots \leq \left(\frac{1}{\mu(B_X(e, \frac{1}{4}))} \int_{B_X(C_0 r + .25)} \left| \sum_{g \in G} (f(g) - c) \chi_g(x) \right|^p dx \right)^{\frac{1}{p}}.$$

Now we rewrite this using the fact that $\sum_{g \in G} \chi_g(x) = 1$.

$$\begin{aligned} \dots &= \left(\frac{1}{\mu(B_X(e, \frac{1}{4}))} \int_{B_X(C_0 r + .25)} \left| \sum_{g \in G} f(g) \chi_g(x) - c \right|^p dx \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{\mu(B_X(e, \frac{1}{4}))} \right)^{\frac{1}{p}} \| \text{comp } f - c \|_{p, B_X(C_0 r + .25)}. \end{aligned}$$

□

We'd also like to compare the norms of the gradients. To do this, we want to write the gradient in such a way that we can compare it with the one on G . We first note that:

$$\nabla \left(\sum_{g \in G} \chi_g(x) \right) = \nabla 1 = 0.$$

This allows us to write the gradient of $\text{comp } f(x)$ as

$$\nabla \text{comp } f(x) = \sum_{g \in G} f(g) \nabla \chi_g(x) = \sum_{g \in G} (f(g) - f(h)) \nabla \chi_g(x).$$

Lemma 4.4.2. *Let $f : G \rightarrow R$ be given. Then for any r we have:*

$$\| \nabla \text{comp } f(x) \|_{p, B_X(r)}^p \leq C \| \nabla f \|_{p, B_G(C_X G(r + 3C_{sup}))}^p$$

where $C = C_g^p \mu(B_X(e, C_{sup})) \text{Vol}_G(G \cap B_X(e, 2C_{sup}))^p |S|^p$.

If we have $B(e, 2C_{sup}) = S$, the generating set, then this is:

$$\|\nabla \text{comp } f(x)\|_{p, B_X(r)}^p \leq C \|\nabla f\|_{p, B_G(C_{XG}(r+C_{sup}))}^p$$

where $C = C_g^p \mu(B_X(e, C_{sup})) |S|^p$.

When $r = \infty$, this is:

$$\|\nabla \text{comp } f(x)\|_{p, X}^p \leq C \|\nabla f\|_{p, G}^p.$$

Proof. We can cover X with balls of radius C_{sup} . This lets us rewrite the norm as follows:

$$\begin{aligned} \|\nabla \text{comp } f(x)\|_{p, B_X(r)}^p &= \int_{B_X(r)} |\nabla \text{comp } f(x)|^p dx \\ &\leq \sum_{h \in G \cap B_X(r+C_{sup})} \int_{B_X(h, C_{sup})} |\nabla \text{comp } f(x)|^p dx \\ &= \sum_{h \in G \cap B_X(r+C_{sup})} \int_{B_X(h, C_{sup})} \left| \sum_{g \in G} (f(g) - f(h)) \nabla \chi_g(x) \right|^p dx \end{aligned}$$

From its definition, we know that $\nabla \chi_g(x)$ will be nonzero only when $d_X(x, g) < C_{sup}$. As we're integrating over x with $d_X(x, h) \leq C_{sup}$, we can restrict our possible g to those with $d_X(g, h) < 2C_{sup}$. Then we use the fact that $|\nabla \chi_g(x)| \leq C_g$.

$$\begin{aligned} &\sum_{h \in G \cap B_X(r+C_{sup})} \int_{B_X(h, C_{sup})} \left| \sum_{g \in B_X(h, 2C_{sup})} (f(g) - f(h)) \nabla \chi_g(x) \right|^p dx \\ &\leq C_g^p \sum_{h \in G \cap B_X(r+C_{sup})} \left| \sum_{g \in G \cap B_X(h, 2C_{sup})} (f(g) - f(h))^p \mu(B_X(h, C_{sup})) \right|. \end{aligned}$$

Note that by invariance, $\mu(B_X(h, C_{sup})) = \mu(B_X(e, C_{sup}))$. At this point, if we had $G \cap B_X(e, 2C_{sup}) = S$, the generating set, we could proceed as follows. Otherwise,

we'll need to expand things a little bit more.

$$\begin{aligned}
& C_g^p \mu(B_X(e, C_{sup})) |S|^p \sum_{h \in G \cap B_X(r+C_{sup})} \left| \sum_{s \in S} \frac{1}{|S|} (f(hs) - f(h)) \right|^p \\
& \leq C_g^p \mu(B_X(e, C_{sup})) |S|^p \sum_{h \in G \cap B_X(r+C_{sup})} \left(\left| \sum_{s \in S} \frac{1}{|S|} (f(hs) - f(h))^2 \right| \right)^{p/2} \\
& = C_g^p \mu(B_X(e, C_{sup})) |S|^p \|\nabla f\|_{p, G \cap B_X(r+C_{sup})}^p \\
& \leq C_g^p \mu(B_X(e, C_{sup})) |S|^p \|\nabla f\|_{p, B_G(C_{XG}(r+C_{sup}))}^p.
\end{aligned}$$

If $G \cap B(e, 2C_{sup}) \neq S$, we could modify this by noting that:

$$\begin{aligned}
& \sum_{h \in G \cap B_X(r+C_{sup})} \left| \sum_{g \in G \cap B_X(h, 2C_{sup})} (f(g) - f(h)) \right|^p \\
& = \sum_{h \in G \cap B_X(r+C_{sup})} \left| \sum_{g \in G \cap B_X(h, 2C_{sup})} \sum_{i=0: s_0 \dots s_k = h^{-1}g}^{k-1} (f(hs_0 \dots s_i) - f(hs_0 \dots s_{i+1})) \right|^p \\
& \leq \sum_{h \in G \cap B_X(r+3C_{sup})} \text{Vol}_G(G \cap B_X(e, 2C_{sup}))^p \sum_{s \in S} (f(hs) - f(h))^p.
\end{aligned}$$

This will yield the inequality:

$$\|\nabla \text{comp } f(x)\|_{p, B_X(r)}^p \leq C \|\nabla f\|_{p, B_G(C_{XG}(r+3C_{sup}))}^p$$

for $C = C_g^p \mu(B_X(e, C_{sup})) \text{Vol}_G(G \cap B_X(e, 2C_{sup}))^p |S|^p$.

Note that in these, C_g is the bound on the gradient of χ_g . □

4.5 Poincaré inequality for volume doubling finitely generated groups

We can use these estimates along with our knowledge of complexes in order to show that volume doubling finitely generated groups admit a strong Poincaré Inequality.

This is not a new fact, but it is a cute proof.

Theorem 4.5.1. *Let G be a finitely generated volume doubling group. Let $f : G \rightarrow \mathbb{R}$ and $B_G(r) \subset G$ be given. Then*

$$\|f - f_{B_G(r)}\|_{1, B_G(r)} \leq Cr \|\nabla f\|_{1, B_G(r)}.$$

Here $C = 4C_P C_g |S| C_{sup}$ where C_P is the constant in the global Poincaré inequality for X .

Proof. Take any such group, and let X be its Cayley graph. Theorem 4.3.2 showed that strong Poincaré inequalities hold on X . We happily note that both C_0 and C_{XG} are 1 on a Cayley graph, and so we can omit them from our calculation. We form a chain of inequalities as follows. From Theorem 4.4.1, we can set $c = (\text{comp } f)_{B_X(r+.25)}$ to get:

$$\begin{aligned} \|f - (\text{comp } f)_{B_X(r+.25)}\|_{1, B_G(r)} \\ \leq \frac{1}{\mu(B_X(g, \frac{1}{4}))} \|\text{comp } f - (\text{comp } f)_{B_X(r+.25)}\|_{1, B_X(r+.25)}. \end{aligned}$$

Note that for every $g \in G$, $\mu(B_X(g, \frac{1}{4})) = 1/4|S|$. From Theorem 4.3.2, we know that:

$$\|\text{comp } f - (\text{comp } f)_{B_X(r+.25)}\|_{1, B_X(r+.25)} \leq C_P r \|\nabla \text{comp } f\|_{1, B_X(r+.25)}.$$

Then we transfer back, using the fact that $G \cap B_X(e, 2C_{sup}) = S$.

$$\|\nabla \text{comp } f(x)\|_{1, B_X(r+.25)} = C_g \mu(B_X(C_{sup})) |S| \|\nabla f\|_{1, B_G(r+.25+C_{sup})}.$$

We can evaluate this as X is a Cayley graph: $\mu(B_X(e, C_{sup})) = |S| C_{sup}$.

Since X is a graph whose edges have unit length, $C_{sup} < 1$. In particular, we can pick $C_{sup} = .74$. Since our original ball, $B_G(r)$ is on the group, without loss of generality we know that r is an integer. Then $B_G(r + C_{sup} + .25) = B_G(r + .99) = B_G(r)$ on the group. Combining this, we have:

$$\|f - \text{comp } f_{B_X(r+.25)}\|_{1, B_G(r)} \leq \frac{1}{.25|S|} C_P r C_g |S| C_{sup} |S| \|\nabla f\|_{1, B_G(r)}.$$

We can get the desired left hand side from

$$\|f - f_{B_G(r)}\|_{1, B_G(r)} \leq \|f - \text{comp } f_{B_X(r+.25)}\|_{1, B_G(r)}.$$

We can use the graph structure to reduce this to:

$$\|f - f_{B_G(r)}\|_{1, B_G(r)} \leq 4C_P C_g |S| C_{sup} r \|\nabla f\|_{1, B_G(r)}.$$

□

Chapter 5

Comparing Heat Kernels on X and G

The main goal of this chapter is to show that for large times, the heat kernel on the group is comparable to the heat kernel on the complex. The comparison was shown for groups and manifolds by Saloff-Coste and Pittet [26].

Notation 5.0.2. To simplify notation, we use p_t for the heat kernel on the group, and h_t when it is on the complex.

On a finitely generated group, the heat kernel can be used to describe a symmetric random walk. This is a walk where from a point $g \in G$, the probability of moving to gs in one step is $\frac{1}{|S|}$ for each generator $s \in S$. The value of the heat kernel on the diagonal, $p_{2n}(e, e)$ gives us the probability of returning to the same point after $2n$ steps. We are interested in this for even numbers of steps because this avoids parity issues. The set-up for these walks can be found in [21].

Definition 5.0.3. We say $f(t) \approx g(t)$ if there exist positive finite constants C_1, C_2, C_3 , and C_4 so that

$$C_1 f(C_2 t) \leq g(t) \leq C_3 f(C_4 t).$$

We will show that the following holds when $t \geq 1$:

$$p_{2\lfloor t \rfloor}(e, e) \approx \sup_{x \in X} h_t(x, x).$$

Note that it doesn't make sense to compare them for small times, since p_t is only defined for integer values of t .

An important notion in this proof is that of amenability.

Definition 5.0.4. A **Følner sequence** is a sequence of finite subsets, $F(i)$, with the following properties:

- (1) For any $g \in G$ there exists i such that $g \in F(i)$,
- (2) $F(i) \subset F(i+1)$, and
- (3) For any finite subset $Q \subset G$, $\lim_{i \rightarrow \infty} \frac{\#(QF(i))}{\#F(i)} = 1$.

Here, $QF(i)$ refers to the set $\{g : g = qf \text{ with } q \in Q, f \in F(i)\}$.

Definition 5.0.5. G is **amenable** if and only if G admits a Følner sequence.

Example 5.0.6. The group of integers, Z , is amenable. Here, the sets $[-i, i]$ form a Følner sequence.

In order to show this, we will split it into two cases. In the first, we look at when G is nonamenable. Here, $p_{2[t]}(e, e) \approx e^{-t}$. Then we will look at when G is amenable. We will first show p_t is approximately less than or equal to h_t , and then we will show the reverse.

5.1 Heat kernels in the nonamenable case

We now look at the behavior of the heat kernel on X and G when G is nonamenable.

We call H_t is the semigroup form for the heat kernel on X . It is related to $h_t(x, y)$ by $H_t f(x) = \int_X f(y) h_t(x, y) dy$. It is also written as $H_t = e^{-t\Delta}$. Alternatively, h_t is called the transition function for H_t . Estimates on norms of functions and their derivatives can give us estimates on $\|H_t\|_{2 \rightarrow 2}$.

Lemma 5.1.1.

$$\|f\|_2 \leq C \|\nabla f\|_2$$

will be true for all $f \in \text{Dom}(\Delta)$ if and only if for all $t > 0$,

$$\|H_t\|_{2 \rightarrow 2} \leq e^{-t/C}.$$

Proof. We will sketch the proof. We can show the forward implication by using integration by parts:

$$\|\nabla f\|_2^2 = \int |\nabla f|^2 = \int f \Delta f = \int \sqrt{\Delta} f \sqrt{\Delta} f = \|\sqrt{\Delta} f\|_2^2.$$

This tells us that for any non-zero $f \in \text{Dom}(\Delta)$, we have:

$$\frac{\|\sqrt{\Delta} f\|_2^2}{\|f\|_2^2} \geq \frac{1}{C}.$$

We can take a square root and then an infimum to get:

$$\inf_{f \neq 0} \frac{\|\sqrt{\Delta} f\|_2}{\|f\|_2} \geq \frac{1}{\sqrt{C}}.$$

This tells us that $\frac{1}{\sqrt{C}}$ is a lower bound on eigenvalues of $\sqrt{\Delta}$. Spectral theory tells us that $\frac{1}{C}$ is a lower bound on eigenvalues of Δ , and $e^{-\frac{t}{C}}$ is an upper bound on eigenvalues of $H_t = e^{-t\Delta}$. This yields

$$\|H_t\|_{2 \rightarrow 2} \leq e^{-t/C}.$$

For the reverse, consider the fact that

$$E(f, f) = \lim_{t \rightarrow 0} \frac{\langle (H_t - I)f, f \rangle}{t} = -\langle \nabla f, \nabla f \rangle.$$

We can use our bound to get:

$$\lim_{t \rightarrow 0} \frac{\langle (H_t - I)f, f \rangle}{t} \leq \lim_{t \rightarrow 0} \frac{\langle (e^{-t/C} - 1)f, f \rangle}{t} = \langle (-1/C)f, f \rangle.$$

Then since

$$E(f, f) = -\langle \nabla f, \nabla f \rangle,$$

we have

$$-\langle \nabla f, \nabla f \rangle \leq \langle (-1/C)f, f \rangle$$

which gives us

$$\|f\|_2 \leq C\|\nabla f\|_2.$$

□

We can transfer between estimates on $\|H_t\|_{2 \rightarrow 2}$ and $h_t(x, y)$. Since the bound on the norm of f will hold for nonamenable groups, we will combine lemmas 5.1.1 and 5.1.2 to get our heat kernel estimates.

Lemma 5.1.2. *If $\|H_t\|_{2 \rightarrow 2}^2 \leq e^{-2t/C}$, then for all $z \in X$ and $t \geq t'$:*

$$h_t(z, z) \leq h_{t'}(z, z)e^{-t/C}.$$

Proof. Apply H_t to $f(y) = \frac{h_s(y, z)}{\|h_s(\cdot, z)\|_2}$. This gives you

$$H_t \frac{h_s(x, z)}{\|h_s(\cdot, z)\|_2} = \int_X \frac{h_s(y, z)}{\|h_s(\cdot, z)\|_2} h_t(x, y) dy = \frac{h_{t+s}(x, z)}{\|h_s(\cdot, z)\|_2}.$$

Our estimate then tells us:

$$\|H_t\|_{2 \rightarrow 2}^2 \geq \int_X \frac{h_{t+s}(x, z)^2}{\|h_s(\cdot, z)\|_2^2} dx = \int_X \frac{h_{t+s}(x, z)h_{t+s}(z, x)}{\|h_s(\cdot, z)\|_2^2} dx = \frac{h_{2t+2s}(z, z)}{\|h_s(\cdot, z)\|_2^2}.$$

Note that

$$\|h_s(\cdot, z)\|_2^2 = \int_X h_s(y, z)^2 dy = h_{2s}(z, z).$$

When we combine this with the inequality for H_t , we have:

$$\frac{h_{2t+2s}(z, z)}{h_{2s}(z, z)} \leq e^{-2t/C}.$$

Fix z and let $u(t) = h_t(z, z)$. This can be written as:

$$u(t+s) \leq u(s)e^{-t/C}.$$

This is equivalent to:

$$\frac{u(t+s) - u(s)}{t} \leq u(s) \frac{e^{-t/C} - 1}{t}.$$

Taking the limit as $t \rightarrow 0^+$ gives us:

$$u'(s) \leq (-1/C)u(s).$$

This gives us the estimate that $u(t) \leq u(t_0)e^{-t/C}$ for any $t \geq t_0$. Rewriting this, we have the long time decay for all $z \in X$ and $t \geq t'$:

$$h_t(z, z) \leq h_{t'}(z, z)e^{-t/C}.$$

□

Note that the converse is essentially true as well. If $h_t(z, z) \leq h_{t'}(z, z)e^{-t/C}$ for $t \geq t'$, then we can construct an upper bound for $\|H_t\|_{2 \rightarrow 2}^2$ whenever $t \geq t'$.

$$\|H_t\|_{2 \rightarrow 2}^2 = \sup_{\|f\|_2=1} \int_X \left(\int_X f(y) h_t(x, y) dy \right)^2 dx.$$

Note that $\int_X f(y) p_t(x, y) \leq \|f\|_2 \|p_t(x, \cdot)\|_2$ holds by Hölder.

$$\begin{aligned} \dots &\leq \sup_{\|f\|_2=1} \int_X \|f\|_2^2 \|p_t(x, \cdot)\|_2^2 dx \\ &= \int_X \|p_t(x, \cdot)\|_2^2 dx \\ &= \int_X \int_X h_t(x, y) h_t(y, x) dy dx \\ &= \int_X h_{2t}(x, x) dx \\ &\leq \left(\int_X h_{t'}(x, x) dx \right) e^{-2t/C}. \end{aligned}$$

This gives us $\|H_t\|_{2 \rightarrow 2} \leq C' e^{-t/C}$ for $t \geq t'$ where $C' = \int_X h_{t'}(x, x) dx$ depends only on X and t' .

In the case where G is not amenable, it is well known that the heat kernel decays exponentially. This result was shown by Kesten [21]. In particular, for any $f \in \text{Dom}(E)$, we know that:

$$\|f\|_{2,G} \leq C_G \|\nabla f\|_{2,G}.$$

We can use averaging to show that this will hold on X as well.

Lemma 5.1.3. *If G is not amenable and $X/G = Y$, then for any $t' > 0$ there exist constants $C_0 = \sup_{y \in Y} h_{t'}(y, y)$ and $C_1 = \sqrt{C(\delta^2 + C_G^2 C(\delta))}$ so that for all $x, y \in X$*

$$h_t(x, y) \leq C_0 e^{-t/C_1}$$

holds for all $t \geq t'$. Note that $C, C(\delta)$ are as in Lemmas 4.2.1 and 4.2.2.

Proof. Applying Lemma 4.2.1 with $p = 2$, $r = \infty$ gives us:

$$\|f\|_{2,X}^2 \leq C (\delta^2 \|\nabla f\|_{2,X}^2 + \|\text{group } f\|_{2,G}^2).$$

The inequality for groups then tells us this is less than

$$\|f\|_{2,X}^2 \leq C (\delta^2 \|\nabla f\|_{2,X}^2 + C_G^2 \|\nabla(\text{group } f)\|_{2,G}^2).$$

We can then bound the gradient in G by the gradient in X using Lemma 4.2.2 with $p = 2$, $r = \infty$:

$$\|f\|_{2,X}^2 \leq C (\delta^2 \|\nabla f\|_{2,X}^2 + C_G^2 C(\delta) \|\nabla f\|_{2,X}^2).$$

Putting this together, we have:

$$\|f\|_{2,X} \leq \sqrt{C(\delta^2 + C_G^2 C(\delta))} \|\nabla f\|_{2,X}.$$

We can apply this with $C_1 = \sqrt{C(\delta^2 + C_G^2 C(\delta))}$ to the first argument to get $\|H_t\|_{2 \rightarrow 2} \leq e^{-t/C_1}$ on our complex, X . Then, apply Lemma 5.1.2 to get the on-diagonal heat kernel bound for any fixed z .

$$h_t(z, z) \leq h_{t'}(z, z)e^{-t/C_1}.$$

Because $X/G = Y$, we can shift z by elements of G , and it won't affect our heat kernel. Specifically, this means $h_t(z, z) = h_t(z + g, z + g)$ for any $g \in G$. This allows us to consider only values of $h_t(y, y)$ for points $y \in Y$. This tells us that the supremum in Y dominates: $\sup_{y \in Y} h_t(y, y) \geq h_t(z, z)$. Set $C_0 = \sup_{y \in Y} h_{t'}(y, y)$. Because Y is compact and $h_{t'}(y, y)$ is continuous in y , for fixed $t' > 0$ we will have $C_0 < \infty$. As $\sup_{x, y} h_t(x, y) = \sup_y h_t(y, y)$, this will give us our overall bound. \square

Corollary 5.1.4. *If G is not amenable and $X/G = Y$, then for $t \geq 1$*

$$\sup_{x \in X} h_t(x, x) \approx p_{2[t]}(e, e).$$

Proof. Kesten [21] showed that nonamenable groups have heat kernel behavior $p_{2[t]}(e, e) \approx e^{-t/C}$. By lemma 5.1.3, we have $h_t(x, x) \leq ce^{-t/C}$. Since $\sup_{x \in X} h_t(x, x) \geq c'e^{-t/C'}$, we have the equivalence. \square

5.2 Heat Kernels in the Amenable Case

This is a modified version of the argument in LSC-Pittet paper [26] which shows that the on diagonal heat kernel on a group is bounded above (in some sense) by the one on a manifold. The basic argument involves comparing eigenvalues and traces of the heat equation restricted to a finite set. We iterate through these sets using Følner sequences, and then we compare the heat kernels themselves.

5.2.1 Bounding those on G above by those on X

Theorem 5.2.1. *Let G be an amenable group and X the associated complex. For times $t > 1$, we have constants C, C_0 so that*

$$p_{\lceil Ct \rceil}(e, e) \leq C_0 \sup_{x \in X} h_t(x, x).$$

Here $C = 2C_1C_2$ where C_1 and C_2 are the constants in 4.4.1 and 4.4.2 and

$$C_0 = |S|^{C_{XG}R_0 / \min_{g \neq h} d_G(g, h)}.$$

Proof. Let A be a finite subset of G , and let A_0 be the set of points in X which surround it. That is, $A_0 := \{x \in X | d(x, A) < R_0\}$.

Because $R_0 \geq C_{sup}$, we will have functions $f : G \rightarrow R$ which are supported in A map to functions $\text{comp } f : X \rightarrow R$ which are supported in A_0 . Using lemmas 4.4.1 and 4.4.2, we know that $\|f\|_2^2 \leq C_1 \|\text{comp } f\|_2^2$ and $\|\nabla \text{comp } f\|_2^2 \leq C_2 E(f, f)$.

Combining these, we get:

$$\begin{aligned} \frac{\|\nabla \text{comp } f\|_2^2}{\|\text{comp } f\|_2^2} &\leq C_1 C_2 \frac{E(f, f)}{\|f\|_2^2} \\ &= C_1 C_2 \frac{\langle I - K_A f, f \rangle}{\|f\|_2^2} \\ &= C_1 C_2 \left(1 - \frac{\|K_A^{1/2} f\|_2^2}{\|f\|_2^2} \right). \end{aligned}$$

Here, we used the fact that K_A is self-adjoint. We can apply the min-max principle in order to compare eigenvalues. Let $\lambda_{A_0}(i)$ be the i th eigenvalue for H_t on $A_0 \subset X$ (denoted $H_t^{A_0}$) and $\beta_A(i)$ i th eigenvalue for K on $A \subset G$ (denoted K_A). For eigenvalues $1..|A|$, we have:

$$\lambda_{A_0}(i) \leq C_1 C_2 (1 - \beta_A(i))$$

We can rewrite this as:

$$\beta_A(i) \leq 1 - \frac{1}{C_1 C_2} \lambda_{A_0}(i).$$

As $\lambda_{A_0}(i)$ will be bounded below by 0, we can use $1 - x \leq e^{-x}$ to get:

$$\beta_A(i) \leq e^{-\frac{1}{c_1 c_2} \lambda_{A_0}(i)}.$$

We can use this to compare the traces. Recall

$$\begin{aligned} \text{Tr}(H_t^{A_0}) &= \sum_i e^{-t \lambda_{A_0}(i)} \\ \text{Tr}(K_A^n) &= \sum_i \beta_A^n(i). \end{aligned}$$

When $\beta_A(i) \geq 0$, we have:

$$\beta_A^{2n}(i) \leq e^{-\frac{1}{c_1 c_2} \lambda_{A_0}(i) 2n}.$$

We will compare the negative $\beta_A(i)$ terms with the positive ones. We know that $0 \leq \text{Tr}(K_A^{2n+1})$. This means we can split the sum into two pieces and subtract the part with negative eigenvalues from both sides:

$$\sum_{\beta_A(i) < 0} |\beta_A^{2n+1}(i)| \leq \sum_{\beta_A(i) > 0} \beta_A^{2n+1}(i).$$

Since all of the eigenvalues are between -1 and 1, we have:

$$\sum_{\beta_A(i) < 0} |\beta_A^{2n+2}(i)| \leq \sum_{\beta_A(i) < 0} |\beta_A^{2n+1}(i)| \leq \sum_{\beta_A(i) > 0} \beta_A^{2n+1}(i) \leq \sum_{\beta_A(i) > 0} \beta_A^{2n}(i).$$

This tells us that

$$\text{Tr}(K_A^{2n+2}) \leq \sum_{\beta_A(i)} |\beta_A^{2n+2}(i)| \leq 2 \sum_{\beta_A(i) > 0} \beta_A^{2n}(i) \leq 2 \sum_i \beta_A^{2n}(i).$$

We can compare the first $|A|$ terms in the two sums, and the extra terms in $\text{Tr}(H_t^{A_0})$ will only help us:

$$\text{Tr}(K_A^{2n+2}) \leq 2 \text{Tr}(H_{\frac{2n}{c_1 c_2}}^{A_0}).$$

We are now in a good spot. We will compare the heat kernels with the respective traces. Fix n . Let $F(i)$ be a Følner sequence in G , and recall S^n is the set of words

in G of length at most n . For each i we will have a set

$A = S^n F(i) = \{g : g = fu, f \in F(i), u \in S^n\}$. In Lsc-Pittet [26], they showed that for an amenable group G we have the comparison:

$$p_{2n+2}(e, e) \leq \frac{1}{|F(i)|} \text{Tr}(K_A^{2n+2}).$$

By the definition of the trace, we know that on the complex we have:

$$\begin{aligned} \text{Tr}(H_t^{A_0}) &= \sum_i e^{-t\lambda_{A_0}(i)} \\ &= \int_{A_0} h_t^{A_0}(x, x) dx \\ &\leq \mu(A_0) \sup_{x \in A_0} h_t^{A_0}(x, x) \\ &\leq \mu(A_0) \sup_{x \in X} h_t(x, x). \end{aligned}$$

When we combine these, we find that:

$$p_{C_1 C_2(2n+2)}(e, e) \leq \frac{\mu(A_0)}{|F(i)|} \sup_{x \in X} h_{2n}(x, x).$$

We can compare $\mu(A_0)$ with $\text{Vol}_G(A)$. Since $A_0 := \{x \in X | d_X(x, A) < R_0\}$, each element in A can expand to at most $|S|^{C_{XG} R_0 / \min_{g \neq h} d_G(g, h)}$ new elements in A_0 .

This tells us:

$$p_{C_1 C_2(2n+2)}(e, e) \leq |S|^{C_{XG} R_0 / \min_{g \neq h} d_G(g, h)} \frac{|A|}{|F(i)|} \sup_{x \in X} h_{2n}(x, x).$$

We can now let i go to infinity; since we have a Følner sequence, $\frac{|A|}{|F(i)|} = \frac{|S^n F(i)|}{|F(i)|}$ will become 1. This leaves us with:

$$p_{C_1 C_2(2n+2)}(e, e) \leq |S|^{C_{XG} R_0 / \min_{g \neq h} d_G(g, h)} \sup_{x \in X} h_{2n}(x, x).$$

□

5.2.2 Bounding those on X above by those on G

We'd like to show the reverse inequality. We will do this using a chain of comparisons. First, we will compare $h_t(x, x)$ with $h_t^W(x, x)$, where h_t^W is the diffusions in an open subset $W \subset X$. Then we will compare eigenvalues of $h_t^W(x, x)$ and $p_t^{W'}(e, e)$, where $p_t^{W'}(e, e)$ represents probability of a random walk restricted to a set $W' \subset G$ returning to the identity, using our bounds on norms and minimax inequalities. Lastly, we use a comparison for $p_t^{W'}(e, e)$ and $p_t(e, e)$. At this point, we will remove some of the dependence on W , and limit away other factors to get the final result.

We would like to look at what happens to diffusions in an open subset $W \subset X$. Let τ be the exit time for this set: $\tau = \inf\{t : t \geq 0, X_t \in W\}$. Then by the strong Markov property we have a restricted heat kernel:

$$h_t^W(x, y) = h_t(x, y) - E^x(h_{t-\tau}(X_\tau, y)1_{\tau \leq t}).$$

Here, X_t is a random variable which at time $t = \tau$ will be the point on ∂W where X_t exits W . The term $h_{t-\tau}(X_\tau, y)$ represents going from the point on the boundary to y in the time $t - \tau$ which is left after exiting W . We take the expected value of this where $X_0 = x$. We can bound the expected value above by the maximum value. Since $E^x(h_{t-\tau}(X_\tau, y)1_{\tau \leq t}) \leq \sup_{0 < s < t} \sup_{z \in \partial W} h_s(z, y)$, we have

$$h_t^W(x, y) \geq h_t(x, y) - \sup_{0 < s < t} \sup_{z \in \partial W} h_s(z, y).$$

We can use this to bound $h_t^W(x, x)$ below for x sufficiently far from ∂W .

Lemma 5.2.2. *There exists a constant C_H so that for all $\varepsilon_1 > 0$ there exists a $a > 0$ so that for all open subsets $W \subset X$ and for all $t \geq 6r_1^2$ we know that*

$$h_t^W(x, x) \geq C_H^{-1} \sup_{y \in X} h_{t-3r_1^2}(y, y) - \varepsilon_1.$$

for all $x \in \{x \in W : d(x, \partial W) > at^{1/2}\}$. Here, $r_1 = \text{diam}(Y)$.

Proof. By Corollary 3.1.5 we know that there are constants C_1 and C_2 so that

$$h_t(x, y) \leq \frac{C_1}{\min(t, 1)^{d/2}} e^{-C_2 \frac{d^2(x, y)}{t}}.$$

This estimate allows us to bound $\sup_{0 < s < t} \sup_{z \in \partial W} h_s(z, y)$ whenever y is at a distance at least $at^{1/2}$ away from the boundary of W . If $s \leq 1$, then

$h_s(z, y) \leq \frac{C_1}{s^{d/2}} e^{-C_2 \frac{a^2 t}{s}}$. This has a maximum at $s = \frac{2a^2 C_2 t}{d}$ which tells us:

$$h_s(z, y) \leq C_1 \left(\frac{d}{2a^2 C_2 t} \right)^{d/2} e^{-d/2}.$$

For $t \geq 6r_1^2$, this is maximized at $t = 6r_1^2$. We have $C_1 \left(\frac{d}{12r_1^2 C_2 e} \right)^{d/2} a^{-d} < \varepsilon_1$ when $a > C_1^{1/d} \left(\frac{d}{12r_1^2 C_2 e} \right)^{1/2} \varepsilon_1^{-1/d}$.

If $t > s > 1$, then the maximum occurs when $s = t$:

$$h_s(x, y) \leq C_1 e^{-C_2 \frac{a^2 t}{s}} \leq C_1 e^{-C_2 a^2}.$$

We know that $C_1 e^{-C_2 a^2} < \varepsilon_1$ whenever $a > \sqrt{\frac{1}{C_2} \ln(\frac{C_1}{\varepsilon_1})}$.

Thus, whenever $a > \max \left(C_1^{1/d} \left(\frac{d}{12r_1^2 C_2 e} \right)^{1/2} \varepsilon_1^{-1/d}, \sqrt{\frac{1}{C_2} \ln(\frac{C_1}{\varepsilon_1})} \right)$ we have

$$\sup_{0 < s < t} \sup_{z \in \partial W} h_s(z, y) < \varepsilon_1.$$

We can bound $h_t(x, x)$ below using a parabolic Harnack inequality. Theorem 3.5 in Sturm [29] uses techniques in Moser [25] to show that Poincaré and volume doubling locally imply a parabolic Harnack inequality. In our situation, we have uniformly bounded constants for both local Poincaré and volume doubling, and so the constant C_H in the Harnack inequality will also be uniform.

In the language of Sturm:

For all $K \geq 1$ and all $\alpha, \beta, \gamma, \delta$ with $0 < \alpha < \beta < \gamma < \delta$ and $0 < \varepsilon < 2$ there exists

a constant $C_H = C_H(Y_1)$ such that for balls $B_{2r}(x) \subset Y_1$ and all T ,

$$\sup_{(s,y) \in Q^-} u(s,y) \leq C_H \inf_{(s,y) \in Q^+} u(s,y)$$

whenever L_T is a uniformly parabolic operator whose associated Dirichlet form is comparable by a factor of K with the original Dirichlet form, and u is a nonnegative local solution of the parabolic equation $(L_T - \frac{\partial}{\partial T})u = 0$ on $Q = (T - \delta r^2, T) \times B_{2r}(x)$. Here $Q^- = (T - \gamma r^2, T - \beta r^2) \times B_{\varepsilon r}(x)$ and $Q^+ = (T - \alpha r^2, T) \times B_{\varepsilon r}(x)$.

We can translate this language to our situation. For us, $L_T = \Delta$, and so there is no T dependence in the operator. This means the Dirichlet form condition will be trivially satisfied when $K = 1$. We also will take $\varepsilon = 1$. We will set $Y_1 = B_{3r_1}$. This is a ball which is large enough so that every equivalence class of $x \in Y$ has a representative in Y_1 , as well as an associated copy of Y in Y_1 . When $t > 6r_1^2$, we can set $T = t + r^2$, $\alpha = 1$, $\beta = 2$, $\gamma = 4$, and $\delta = 5$. Then $Q^+ = (T - r^2, T + r^2)$, $Q^- = (T - 4r^2, T - 2r^2)$, and $Q = (T - 5r^2, T + r^2)$. Applying Sturm here gives us

$$\sup_{y \in B_r(x)} h_{t-3r^2}(y,y) \leq C_H \inf_{y \in B_r(x)} h_t(y,y) \leq C_H h_t(x,x).$$

Due to the symmetry of the space X , $h_s(y,y)$ is the same as h_s when y is translated by an element of G . For $r = \text{diam}(Y)$, we have a copy of $Y \subset B_r(x) \subset B_{2r}(x) \subset Y_1$ for every $x \in Y$. This tells us that

$$\sup_{y \in B_r(x)} h_s(y,y) = \sup_{y \in X} h_s(y,y).$$

□

We can bound the integral of $h_t^W(x,x)$ above by an analogue of Lemma 5.3 in LSC-Pittet [26].

Lemma 5.2.3. *For subsets $W \subset X$, $B > 0$, and $t \geq 1$,*

$$\int_W h_t^W(x, x) dx \leq \sum_{\lambda_W(i) \leq 1/B} e^{-t\lambda_W(i)} + C_1 2^{d/2} \mu(W) e^{-t/(2B)}.$$

Here, C_1 and d are defined as in Corollary 3.1.5, and λ_W are the eigenvalues of h_t^W .

Proof. When $a, b \geq 1$ we have the inequality $ab \geq a/2 + b/2$. Let $a = t$ and $b = B/\lambda_W(i)$. Then for $t \geq 1$ and $\lambda_W(i) \geq 1/B$ we have

$$tB/\lambda_W(i) \geq t/2 + B/(2\lambda_W(i)).$$

If we multiply through by $-1/B$ and exponentiate we find

$$e^{-t\lambda_W(i)} \leq e^{-t/(2B) - \lambda_W(i)/2}.$$

This allows us to bound the sum over the larger eigenvalues:

$$\begin{aligned} \sum_{\lambda_W(i) \geq 1/B} e^{-t\lambda_W(i)} &\leq \sum_{\lambda_W(i) \geq 1/B} e^{-t/(2B) - \lambda_W(i)/2} \\ &\leq e^{-t/(2B)} \sum_{\lambda_W(i)} e^{-\lambda_W(i)/2} \\ &= e^{-t/(2B)} \int_W h_{1/2}^W(x, x) dx \\ &\leq e^{-t/(2B)} C_1 2^{d/2} \mu(W). \end{aligned}$$

In the last step, we used the bound in 3.1.5 which tells us $h_{1/2}^W(x, x) \leq C_1 2^{d/2}$.

Using the eigenvalue expansion, we can compare the integral of the heat kernel at times greater than one with the sum over small eigenvalues plus our bound on the sum over larger eigenvalues:

$$\begin{aligned} \int_W h_t^W(x, x) dx &= \sum_{\lambda_W(i)} e^{-t\lambda_W(i)} \\ &\leq \sum_{\lambda_W(i) \leq 1/B} e^{-t\lambda_W(i)} + C_1 2^{d/2} \mu(W) e^{-t/(2B)}. \end{aligned}$$

□

Let's consider what it means to have a Laplacian, Δ^Ω , defined for functions restricted to a set, Ω with a polygonal boundary. Let the domain of Δ^Ω be the closure of the intersection of $\text{Dom}(\Delta)$ and the continuous functions which are compactly supported on Ω ; that is, $\text{Dom}(\Delta^\Omega) = \overline{\text{Dom}(\Delta) \cap C_0^C(\Omega)}$. Note that since $\text{Dom}(\Delta) \cap C_0(\Omega) \subset \text{Dom}(\Delta)$ and $\text{Dom}(\Delta)$ is closed, we know that $\text{Dom}(\Delta^\Omega) \subset \text{Dom}(\Delta)$.

For functions $f \in \text{Dom}(\Delta^\Omega)$, we set $\Delta^\Omega f = \Delta f$. Δ^Ω inherits many properties from Δ . It is self-adjoint with a discrete spectrum, and as we will see in the following lemma, for the Ω that we are interested in there will be only finitely many eigenvalues which are close to 0.

We can show this by comparing operators restricted to subsets of X to operators restricted to subsets of G . Let $A \subset G$ be given. Let $\Omega = U(A)$ be a subset of X with polygonal boundary so that any function f whose support is in $U(A)$ has an associated function group f whose support is in $(A, \{1..N\})$. In particular, we would like $U(A)$ to be close in size to A . Since group $f(g, i) = \int_{B_X(g\gamma_i, \delta)} f(x) dx$ averages over neighborhoods of points in X , we can guarantee a set with volume estimate:

$$\min_{y \in Y} \mu(B(y, \delta)) N \# A \leq \mu(U(A)) \leq \mu(Y) N \# A.$$

The following lemma will give us a comparison for small eigenvalues on $h_t^{U(A)}$.

Lemma 5.2.4. *Let $A \subset G$ and $U(A) \subset X$ be given as above. Eigenvalues of $h_t^{U(A)}(x, x)$ and $p_n(e, e)$ are comparable in the following manner:*

$$\sum_{i: 0 \leq \lambda_{U(A)}(i) \leq 1/B} e^{-2n\lambda_{U(A)}(i)} \leq \# A N p_{2\lfloor n/(B(2-\sqrt{2})) \rfloor}(e, e).$$

Here $B = 4C(\sqrt{\frac{1}{2C_{grad}}})/(2 - \sqrt{2})$ where C_1 is the constant in Lemma 4.2.1 and $C(\cdot)$ is the constant from Lemma 4.2.2. N depends on $\delta = \sqrt{\frac{1}{2C_{grad}}}$.

Proof. Suppose u is a solution to $\Delta^\Omega u = \lambda u$ on a set $\Omega \subset X$ with polygonal boundary, and $u = 0$ on $\partial\Omega$. Set $u = 0$ outside of Ω . For $u \in \text{Dom}(\Delta^\Omega)$ and $\lambda \neq 0$, a formal argument using integration by parts tells us:

$$\begin{aligned} \langle u, u \rangle &= \frac{1}{\lambda} \langle \lambda u, u \rangle \\ &= \frac{1}{\lambda} \langle \Delta^\Omega u, u \rangle \\ &= \frac{1}{\lambda} \langle \nabla u, \nabla u \rangle + \langle \nabla u, u \rangle|_{\partial\Omega} \\ &= \frac{1}{\lambda} \langle \nabla u, \nabla u \rangle. \end{aligned}$$

This gives us $\|u\|_2 = |\frac{1}{\lambda}| \|\nabla u\|_2$. We know that such eigenfunctions exist because Δ^Ω is self-adjoint.

We will combine this with the inequality in Lemma 4.2.1 for eigenfunctions f on the set $U(A)$:

$$\begin{aligned} \|f\|_{2,X}^2 &\leq C_{grad}(\delta^2 \|\nabla f\|_{2,X}^2 + \|\text{group } f\|_{2,G}^2) \\ &= C_{grad}(\delta^2 |\lambda| \|f\|_{2,X}^2 + \|\text{group } f\|_{2,G}^2). \end{aligned}$$

This tells us:

$$(1 - C_{grad}\delta^2|\lambda|)\|f\|_{2,X}^2 \leq \|\text{group } f\|_{2,G}^2.$$

If δ is less than $\sqrt{\frac{1}{\lambda C_{grad}}}$, we have a nice bound for that λ . In particular, $\delta = \sqrt{\frac{1}{2C_{grad}}}$ gives us a simple bound for all $\lambda \leq 1$ because $1 - (1/2)|\lambda| > 1/2$.

$$\|f\|_{2,X}^2 \leq 2\|\text{group } f\|_{2,G}^2.$$

Lemma 4.2.2 tells us

$$\|\nabla \text{group } f\|_{2,G}^2 \leq C(\delta) \|\nabla f\|_{2,X}^2.$$

We have that for $C' = 2C(\sqrt{\frac{1}{2C_{grad}}})$:

$$\frac{\|\nabla \text{group } f\|_{2,G}^2}{\|\text{group } f\|_{2,G}^2} \leq C' \frac{\|\nabla f\|_{2,X}^2}{\|f\|_{2,X}^2}.$$

We can rewrite $\nabla \text{group } f$ in terms of $K_{A,N}^{1/2}$.

$$1 - \frac{\|K_{A,N}^{1/2} \text{group } f\|_{2,G}^2}{\|\text{group } f\|_{2,G}^2} \leq C' \frac{\|\nabla f\|_{2,X}^2}{\|f\|_{2,X}^2}.$$

This will allow us to compare the first k eigenvalues of $h_t^{U(A)}$ with the absolute values of those for $K_{A,N}$, where $k = \min(\#NA, \#\{\lambda_{U(A)}(i) \in [0, 1]\})$. The min-max definition will give us these eigenvalue comparisons. For simplicity, we will use $\lambda_{U(A)}(i)$ to refer to the i th smallest eigenvalue of $h_t^{U(A)}$, and $|\beta_A(i)|$ to refer to the i th largest absolute value of the eigenvalue of $K_{A,N}$. We have

$$1 - |\beta_A(i)| \leq C' \lambda_{U(A)}(i) \text{ which can be written as}$$

$$1 - C' \lambda_{U(A)}(i) \leq |\beta_A(i)|.$$

When $1/2 \leq x \leq 1$, we know $x \geq e^{-2(1-x)}$. Applying that to $x = 1 - C' \lambda_{U(A)}(i)$, we have

$$e^{-2C' \lambda_{U(A)}(i)} \leq |\beta_A(i)|$$

for $i \leq k$ with $0 \leq \lambda_{U(A)}(i) \leq 1/(2C')$. We can exponentiate to get:

$$e^{-2n \lambda_{U(A)}(i)} \leq |\beta_A(i)|^{n/C'}.$$

We will have this bound for all of the $\lambda_{U(A)} \in [0, (2 - \sqrt{2})/(2C')]$ provided we can show that we have an i with $C' \lambda_{U(A)}(i) > (2 - \sqrt{2})/2$. If we knew that $(2 - \sqrt{2})/2 \leq 1 - |\beta_A(i)|$ for some i , then this would be shown. This means we want to have $|\beta_A(i)|^2 \leq 1/2$ for some i . We know that $K_{A,N}$ is an $\#AN$ by $\#AN$ matrix whose entries are either $1/|S|$ or 0 and that there are $|S|$ nonzero entries

per row. When we look at its square, we have another $\#AN$ by $\#AN$ matrix whose entries are at most $|S|/|S|^2 = 1/|S|$ and at least 0. $K_{A,N}^2$ has eigenvalues $|\beta_A(i)|^2$. This means that the largest $\text{Tr}(K_{A,N}^2)$ could possibly be is $\#AN/|S|$, and so $\sum_{i=1}^{\#AN} |\beta_A(i)|^2 \leq \#AN/|S|$. The average value of an eigenvalue $|\beta_A|^2$ is $1/|S|$. Since $|\beta_A(i)|^2 \in [0, 1]$, we must have at least one $|\beta_A|^2$ which is smaller than $1/|S|$ in order to have that as the average. This tells us that there is some i with $|\beta_A(i)| \leq 1/\sqrt{|S|} \leq 1/\sqrt{2}$.

In this way, we have guaranteed the bound for all $\lambda_{U(A)} \in [0, (2 - \sqrt{2})/(2C')]$. Note that this also shows that there are at most $\#AN$ such eigenvalues.

Summing over $\lambda_{U(A)}(i) \in [0, (2 - \sqrt{2})/(2C')]$ gives us:

$$\sum_{i: 0 \leq \lambda_{U(A)}(i) \leq (2 - \sqrt{2})/(2C')} e^{-2n\lambda_{U(A)}(i)} \leq \sum_{i: 0 \leq \lambda_{U(A)}(i) \leq (2 - \sqrt{2})/(2C')} (\beta_A(i))^{n/C'}.$$

Note that the $\beta_A(i)$ in this sum are positive. We can compare these to positive eigenvalues in the trace by using $K_{A,N}^{2\lfloor n/(2C') \rfloor}$.

$$\sum_{i: 0 \leq \lambda_{U(A)}(i) \leq (2 - \sqrt{2})/(2C')} (\beta_A(i))^{n/C'} \leq \text{Tr}(K_{A,N}^{2\lfloor n/(2C') \rfloor}).$$

Combining these yields:

$$\sum_{i: 0 \leq \lambda_{U(A)}(i) \leq (2 - \sqrt{2})/(2C')} e^{-2n\lambda_{U(A)}(i)} \leq \text{Tr}(K_{A,N}^{2\lfloor n/(2C') \rfloor}).$$

We know that by its definition

$$\begin{aligned} \text{Tr}(K_{A,N}^{2\lfloor n/(2C') \rfloor}) &= \sum_{g \in A, j=1..n} p_{2\lfloor n/(2C') \rfloor}(g, g) \\ &\leq \#AN p_{2\lfloor n/(2C') \rfloor}(e, e). \end{aligned}$$

This gives us the result:

$$\sum_{i: 0 \leq \lambda_{U(A)}(i) \leq (2 - \sqrt{2})/(2C')} e^{-2n\lambda_{U(A)}(i)} \leq \#AN p_{2\lfloor n/(2C') \rfloor}(e, e).$$

If we want to simplify the notation on the left, we may set $B = 2C'/(2 - \sqrt{2})$.

This means $n/(2C') = n/(B(2 - \sqrt{2}))$. Hence:

$$\sum_{i: 0 \leq \lambda_{U(A)}(i) \leq 1/B} e^{-2n\lambda_{U(A)}(i)} \leq \#ANp_{2\lfloor n/(B(2-\sqrt{2})) \rfloor}(e, e).$$

□

Theorem 5.2.5. *For $t > 6r_1^2$ we get:*

$$\sup_{y \in X} h_{t-3r_1^2}(y, y) \leq Cp_{2\lfloor \frac{t}{B \log |S|} \rfloor}(e, e)$$

$$\text{where } C = C_H \left(\frac{1}{\min_{y \in Y} \mu(B(y, \delta))} + \frac{\mu(Y)}{\mu(B(y, \delta))} C_1 2^{d/2} \right).$$

Proof. We'll use these lemmas and Følner sequences to build this inequality.

Recall Lemma 5.2.2 told us:

$$h_t^W(x, x) \geq C_H^{-1} \sup_{y \in X} h_{t-3r_1^2}(y, y) - \varepsilon_1.$$

for all $x \in \{x \in W : d(x, \partial W) > at^{1/2}\}$ when $t > 6r_1^2$.

Set $T = \{g \in G : d_X(e, g) \leq \sqrt{t}a + 10R_0\}$.

Then $AT = \{g \in G : g = th \text{ for } t \in T, h \in A\}$. We'll apply this to $W = U(AT)$.

Note that $U(A) \subset \{x \in U(AT) : d(x, \partial U(AT)) > at^{1/2}\}$.

When we take the average over $U(A)$ we have:

$$\sup_{y \in X} h_{t-3r_1^2}(y, y) \leq C_H \left(\int_{U(A)} h_t^{U(AT)}(x, x) dx + \varepsilon_1 \right)$$

From Lemma 5.2.3 we know how to bound the integral in terms of $\lambda_W \leq 1/B$:

$$\begin{aligned} \int_{U(A)} h_t^{U(AT)}(x, x) dx &\leq \int_{U(AT)} h_t^{U(AT)}(x, x) dx \\ &\leq \sum_{\lambda_{U(AT)}(i) \leq 1/B} e^{-t\lambda_{U(AT)}(i)} + C_1 2^{d/2} \mu(U(AT)) e^{-t/(2B)}. \end{aligned}$$

Putting them together gives us:

$$\begin{aligned} & \sup_{y \in X} h_{t-3r_1^2}(y, y) \\ & \leq C_H \left(\frac{1}{\mu(U(A))} \sum_{\lambda_{U(AT)}(i) \leq 1/B} e^{-t\lambda_{U(AT)}(i)} + \frac{\mu(U(AT))}{\mu(U(A))} C_1 2^{d/2} e^{-t/(2B)} + \varepsilon_1 \right). \end{aligned}$$

By Lemma 5.2.4 we have:

$$\sum_{i: 0 \leq \lambda_{U(AT)}(i) \leq 1/B} e^{-2n\lambda_{U(AT)}(i)} \leq \#(AT) N p_{2\lfloor \frac{n}{B(2-\sqrt{2})} \rfloor}(e, e).$$

When we set $n = t/2$, this gives us:

$$\begin{aligned} & \sup_{y \in X} h_{t-3r_1^2}(y, y) \\ & \leq C_H \left(\frac{\#(AT)N}{\mu(U(A))} p_{2\lfloor \frac{t}{2B(2-\sqrt{2})} \rfloor}(e, e) + \frac{\mu(U(AT))}{\mu(U(A))} C_1 2^{d/2} e^{-t/(2B)} + \varepsilon_1 \right). \end{aligned}$$

On G , we can bound below the probability of returning to the start by noting that because $S = S^{-1}$, after moving n steps, we have a $\frac{1}{|S|^n}$ chance of exactly retracing our path.

$$p_{2n}(e, e) \geq \frac{1}{|S|^n} = e^{-n \log |S|}.$$

A more convenient time gives us

$$p_{2\lfloor \frac{n}{2B \log |S|} \rfloor}(e, e) \geq e^{-n/(2B)}.$$

When we place this into the inequality, we have:

$$\begin{aligned} & \sup_{y \in X} h_{t-3r_1^2}(y, y) \\ & \leq C_H \left(\frac{\#(AT)N}{\mu(U(A))} p_{2\lfloor \frac{t}{2B(2-\sqrt{2})} \rfloor}(e, e) + \frac{\mu(U(AT))}{\mu(U(A))} C_1 2^{d/2} p_{2\lfloor \frac{t}{2B \log |S|} \rfloor}(e, e) + \varepsilon_1 \right). \end{aligned}$$

We can use the fact that $p_t(e, e) \leq p_s(e, e)$ whenever $t > s$ noting that both $2\lfloor \frac{t}{2B(2-\sqrt{2})} \rfloor$ and $2\lfloor \frac{t}{2B \log |S|} \rfloor$ are larger than $2\lfloor \frac{t}{B \log |S|} \rfloor$.

$$\dots \leq C_H \left(\left(\frac{\#(AT)N}{\mu(U(A))} + \frac{\mu(U(AT))}{\mu(U(A))} C_1 2^{d/2} \right) p_{2\lfloor \frac{t}{B \log |S|} \rfloor}(e, e) + \varepsilon_1 \right).$$

We take a Følner sequence for G , and set $A = F(i)$. We can use our volume estimates to find:

$$\frac{\#(AT)N}{\mu(U(A))} \leq \frac{\#(AT)N}{\min_{y \in Y} \mu(B(y, \delta))N\#A} = \frac{\#(AT)}{\#A \min_{y \in Y} \mu(B(y, \delta))}$$

and

$$\frac{\mu(U(AT))}{\mu(U(A))} \leq \frac{\mu(Y)N\#(AT)}{\min_{y \in Y} \mu(B(y, \delta))N\#A} = \frac{\#(AT)\mu(Y)}{\#A \min_{y \in Y} \mu(B(y, \delta))}.$$

When we take the limit of $\frac{\#(AT)}{\#A} = \frac{\#(F(i)T)}{\#F(i)}$ as $i \rightarrow \infty$, we find it is 1.

This gives us:

$$\sup_{y \in X} h_{t-3r_1^2}(y, y) \leq C_H \frac{1 + \mu(Y)}{\min_{y \in Y} \mu(B(y, \delta))} C_1 2^{d/2} p_{2\lfloor \frac{t}{B \log |S|} \rfloor}(e, e) + C_H \varepsilon_1.$$

Now let ε_1 go to zero. This yields the comparison. \square

We can combine these three results into a single theorem.

Theorem 5.2.6. *Let G be a finitely generated group and X the associated complex.*

For times $t > 1$, we have the comparison

$$p_{2\lceil t \rceil}(e, e) \approx \sup_{x \in X} h_t(x, x).$$

Note that by transitivity, this holds for the heat kernels on the skeletons as well.

Proof. If G is amenable, apply theorems 5.2.1 and 5.2.5. If G is nonamenable, apply corollary 5.1.4. \square

This theorem gives a comparison of heat kernel behavior at large times. It does not; however, tell you what that behavior is for a given group. Even though the proof tells you the asymptotic for nonamenable groups, it is not easy to determine amenability. For example, it is unknown whether Thompson's group F is amenable or not. (See Belk [2].)

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